# Lower Bounds on Block-Diagonal SDP Relaxations for the Clique Number of the Paley Graphs

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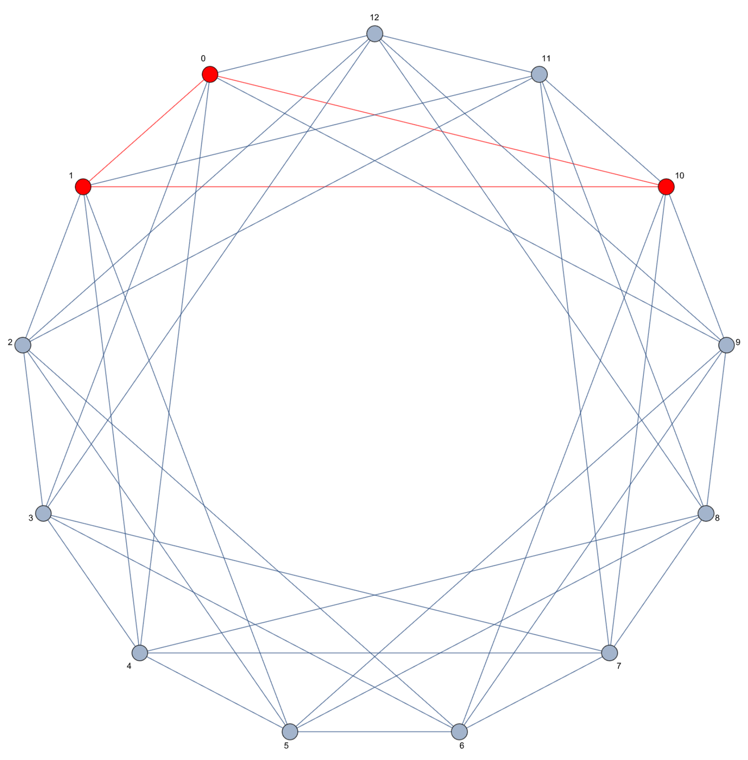
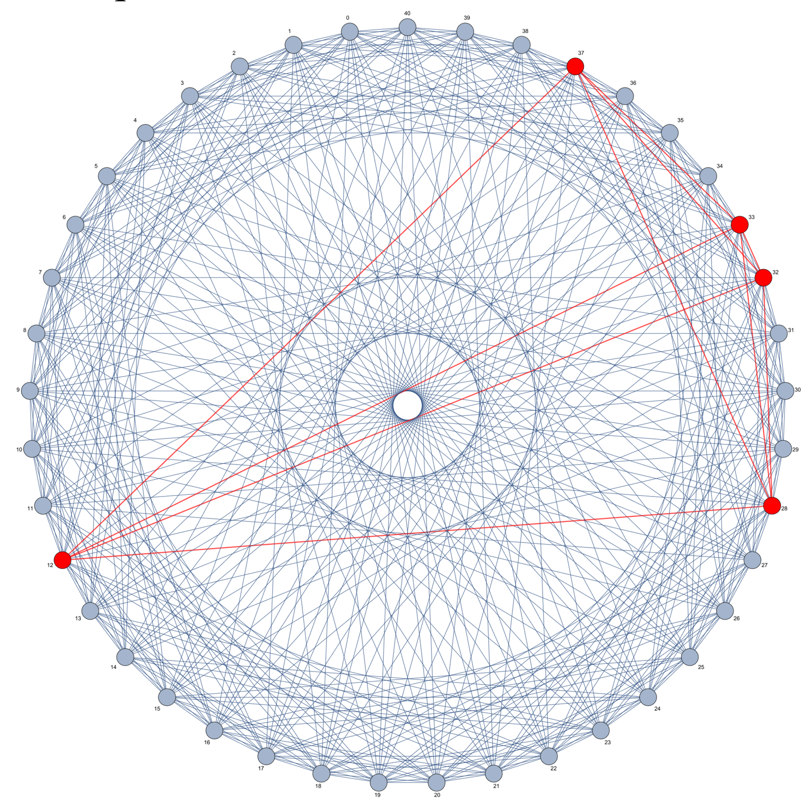
## Maximal clique of the Paley graphs

- *Paley graph* $G_p$, for prime $p \equiv 1 \bmod 4$, is a graph with the vertex set $V(G_p) = \mathbb{F}_p$, where two vertices are connected if their difference is a quadratic residue in $\mathbb{F}_p$
- A subset of vertices in a graph $G$ forms a *clique* if every pair in that subset is adjacent. Write $\mathcal{K}(G)$ as the set of all cliques in $G$.
- The *clique number* $\omega(G)$ is the size of a largest clique in $G$
- For a clique $K$ in $G$, the localization $G_K$ is the subgraph of $G$ induced on all vertices (excluding those in $K$) that are adjacent to every vertex in $K$.
- *Examples*: $G_{13}$ and $G_{41}$ below with the largest cliques marked in red





## Background and Motivations

- Determining $\omega(G)$ of $G = (V, E)$, and even approximating it up to an $O(|V|^{1-\epsilon})$ factor, is a classic NP-hard problem
- Paley graphs $G_p$ in some aspects, *pseudorandom*, behave like the *Erdos-Renyi (ER) graphs* $\mathcal{G}(1/2, p)$ (where $\mathbb{E}[\omega(\mathcal{G}(1/2, p)] \sim 2 \log p$)
- $G_p$'s are conjectured to lead to deterministic restricted isometries in compressed sensing
- The spectral u.b. and Lovasz $\theta$ / $SOS_2$ yield $\omega(G_p) \leq \sqrt{p}$
- The SOTA u.b. by Hanson and Petridis [5] improves on the above by a constant prefactor

$$\omega(G_p) \leq HP(G_p) \sim \frac{\sqrt{p}}{\sqrt{2}}$$

- The SOTA l. b. is

$$\omega(G_p) \geq \log p \log \log \log p$$

- Numerical evidence supports *conjectured* $\omega(G_p) = O(\text{polylog}(p))$
- Achieving an $O\left(p^{\frac{1}{2}-\epsilon}\right)$ u.b. for $\epsilon > 0$, i.e., breaking the so-called $\sqrt{p}$ *bottleneck* for $\omega(G_p)$ is regarded as a difficult open problem in additive combinatorics and TCS
- For random graphs $G \sim \mathcal{G}(\frac{1}{2}, n)$, convex relaxations (Lovasz-Schriver and SOS hierarchies) do *not* break the $\sqrt{n}$ bottleneck
- However, numerical evidence suggests that $SOS_4(G_p)$ and the block diagonal $L^3(G_p)$ relaxation of [4] [1, 2] do break this bottleneck
- The values of the block-diagonal $L^2$ relaxation bound from above the SOS-4 values
- Therefore, the $\Omega(p^{1/3})$ l.b. for $SOS_4$ in [1] also applies to the $L^2$ relaxations
- Thus, it was previously unclear if the $L^2$ relaxation, which is sandwiched between $SOS_2$ and $SOS_4$ may break the $\sqrt{p}$ bottleneck

## Integer program and SDP relaxations

- Given a graph $G = (V, E)$, with $n = |V|$:

$$\omega(G) \ = \ \max \sum_{i \in V} x_i, \quad \text{s.t.} \quad x \in \mathbb{R}^n, \ x_i^2 = x_i \ \ \forall i \in V, \ x_i x_j = 0 \ \ \forall \{i, j\} \notin E$$

- For a vector $y \in \mathbb{R}^{\mathcal{P}_{2t}}$ and $I, J$ in the *power set* $\mathcal{P}_{2t}(V)$ of $V$ with $2t$ elements, the *moment matrix* $M_t(y)$ is given by $M_t(y)_{I,J} = y_{I \cup J}$
- The $SOS_{2t}$ relaxation of the maximal clique number of $G$, is given by:

$$SOS_{2t}(G) \quad := \quad \max \ \sum_{i \in V} y_{i,\varnothing} \ \ \text{s.t.} \ \ y \in \mathbb{R}^{\mathcal{P}_{2t}}, \text{ with } y_\varnothing = 1, \ y_{S,T} = 0 \ \ \forall S \cup T \notin K, \ M_t(y) \succeq 0$$

where $K$ is the set of all cliques of $G$

- The block-diagonal hierarchy $L^t$ further relaxed $SOS_{2t}$ by replacing $M_t(y) \succeq 0$ with PSD conditions for principal submatrices of $M_t(y)$ indexed by

$$A(T) := \bigcup_{S \subseteq T} A_S, \ \text{ where } A_S := \{S\} \cup \{S \cup \{i\} \mid i \in V\}.$$

- For any $G$, $L^t$ for it's clique number problem is

$$L^t(\overline{G}) := \begin{Bmatrix} \max \displaystyle\sum_{i \in V} y_{\{i\}} \\ \text{s.t. } y \in \mathbb{R}^{\mathcal{P}_{t+1}}, \ y_\varnothing = 1, \ y_{\{i,j\}} = 0, \forall \{i, j\} \notin E \\ A(S, T)(y) \succeq 0 \text{ for all } S \subseteq T \text{ and } T \in \mathcal{P}_{=t-1} \end{Bmatrix}$$

where

$$A(S, T)(y) := \sum_{S' : S \subseteq S' \subseteq T} (-1)^{|S' \setminus S|} A_{S'}(y)$$

with

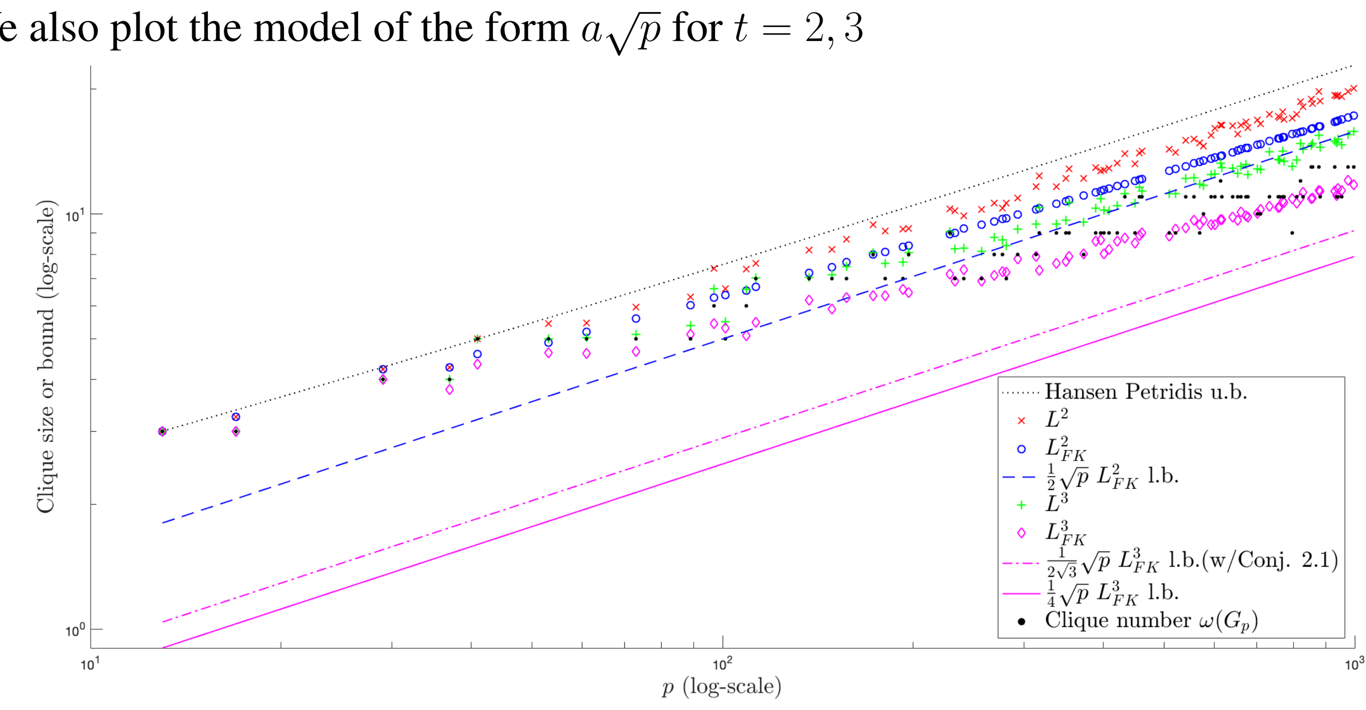
$$A_S(y)_{\varnothing,\varnothing} = y_S, \ \ A_S(y)_{\varnothing,i} = y_{S \cup \{i\}}, \ \ A_S(y)_{i,j} = y_{S \cup \{i,j\}} \ \ (i, j \in V, \text{ where } |V| = n)$$

## Main result: Lower bound on $L^t(G_{p,K})$

- We proved the following lower bound on $L^t(G_{p,K})$ for any $t$ and $K$ of arbitrary size $a := |K|$:

$$L^t(\overline{G}_{p,K}) \geq \frac{\sqrt{p}}{2^{a+t-1}} + O\left(\frac{a}{2^t}\right)$$

- This shows for any fixed $t$ and $a$, $L^t$ and Lovász-Shrijver does not break $\sqrt{p}$ bottleneck.
- However, our bound leaves open the possibility (supported by numerical evidence) that $L^t$ could improve the constant prefactor relative to the $HP(G_p)$ SOTA u.b.
- Since the lower bound is a function of $a + t$, it's consistent with the relaxation-localization trade-off conjectured in [3].
- We also plot the model of the form $a\sqrt{p}$ for $t = 2, 3$




## Proof techniques

- We construct a feasible point of $L^t(G_{p,K})$ using *Feige-Krauthgamer (FK) pseudomoments*, similar to such construction in [1] for $SOS_{2t}$, restricting our attention to

$$\alpha_{|S|} := \mathbb{1}_{S \in \mathcal{K}(G_{p,K})} y_S$$

where $\alpha_0 = 1$

- After removing duplicated rows and columns and possibly removing/padding zero columns, it's enough to consider $\hat{A}(S, T)(y)$, which is the first two levels of the alternating sum in $A(S, T)(y)$. This allows the positive-definiteness condition to be tractable
- For each $\hat{A}(S, T)(\alpha)$, the PSDness is proved by considering it's Schur complement $D_S$
- The minimum eigenvalue $\lambda_{\min}(D_S)$ of $D_S$ can be lower-bounded by analyzing the matrices appeared in the matrices in the first two levels of the alternating sum in $A(S, T)(y)$, using decomposition techniques and characteristic sum estimates
- We assume $\alpha_i = c_i p^{-i/p}$ and proved that by choosing

$$2 > \frac{2\sqrt{p}}{\sqrt{p}+1} \geq \frac{c_{t+1}}{c_t} = \frac{2c_t}{c_{t-1}} = \frac{4c_{t-1}}{c_{t-2}} = \ldots = \frac{2^{t-1}c_2}{c_1} = \frac{2^t c_1}{c_0} = 2^t c_1$$

we have a lower bound for $\lambda_{\min}(D_S)$, which is non-negative

- Therefore, the condition $\hat{A}(S, T)(\alpha) \succeq 0$ holds for any $\varnothing \subseteq S \subseteq T \in \mathcal{P}_{=t-1}$. Showing our choice of $\alpha_i$ leads to a feasible point to $L^t(G_{p,K})$
- These feasible $c_i$'s lead to the lower bound specified above, with an error term of $O(\frac{a}{2^t})$
- For $t = 1$ and certain $a$'s, the above error can be more accurately computed

## Extensions and future works

- Our result also leave open the possibility that the block-diagonal relaxations may improve the constant prefactor of the Hanson-Petridis upper bound
- Such upper bounds may be obtained by constructing feasible points of the corresponding dual programs
- Rather than considering fixed $a, t$, let $a, t$ be slowly growing functions of $p$, say $\epsilon \log(p)$, it could be possible that $L^t(G_{p,K})$ still break the $\sqrt{p}$-barrier

## Acknowledgements & References

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