Lower Bound on the Block-Diagonal SDP Relaxation for the Clique Number of the Paley Graph

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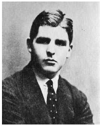
Outline

- The clique number problem and the Paley graphs
- Compressed sensing and sparse recovery motivations
- Block-diagonal (L^t), SOS and Lovasz-Schrivjer SDPs
- Our contributions
  - L^t lower bounds via FK pseudomoments
  - Localization lower bounds, and relaxation-localization trade-off
- Conclusion and future work

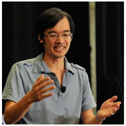
Paley graph clique number

- Classic problem in number theory and additive combinatorics
- Connected to Ramsey theory, random matrices, computational complexity and optimization, to name a few research areas
- Links to deterministic restricted isometries in compressed sensing and sparse recovery



Paley

Ramsey



Tao

Vallentin

Laurent

Gvozdenovic

Background

For any $G = (V, E)$:

- $K \subseteq V$ is a *clique* if each $i, j \in K$ are adjacent
- *Clique number*

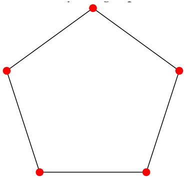
$$\omega(G) = \text{the size of a largest clique}$$

- $I \subseteq V$ is an *independence set* if each $i, j \in I$ are not adjacent
- *Independence number*

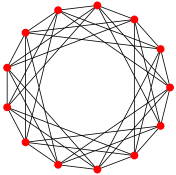
$$\alpha(G) = \text{the size of a largest independence set}$$

Finding $\omega(G)$ and $\alpha(G)$ is NP-hard for general graphs

Paley graph



$\omega(G_5) = 2$



$\omega(G_{13}) = 3$

Image credit: Wolfram

- A *Paley graph* $G_p = (V, E)$
  - $|V| = p$ where $p \equiv 1 \mod 4$ (Pythagorean prime)
  - $\{i, j\} \in E$ iff $i - j = a^2 \mod p$ for some $a \in \mathbb{Z}_p$
  - Strongly regular and self-complementary
- (We're not considering Paley graphs of prime power order p^s)

Connections to compressed sensing and sparse recovery

- SLOGAN: **compressible high-dimensional signal can be recovered from very few measurements**
- $x \in \mathbb{R}^n$ is s-sparse if it has no more than s nonzero entries
- When can you recover x exactly from few measurements y
- Sparse recovery experiment design of $A \in \mathbb{C}^{m \times n}$

$$\begin{bmatrix} y \end{bmatrix} = \begin{bmatrix} \quad & A & \quad \end{bmatrix} \begin{bmatrix} \\ x \\ \\ \end{bmatrix}$$

where

$$s < m \ll n$$

Restricted isometry property (RIP)

- Guarantees that sparse recovery is robust to noise
- $A \in \mathbb{C}^{m \times n}$ satisfies RIP with distortion $0 < \delta < 1$ if for any s-sparse x

$$(1-\delta)\|x\|^2 \leq \|Ax\|_2^2 \leq (1+\delta)\|x\|^2$$

- Matrices with Gaussian i.i.d. entries satisfy RIP w.h.p. if

$$s \sim m/\log(n)$$

Square root bottleneck

- Deterministic constructions based on controlling "spikeness" or "localization" (coherence) of rows achieve

$$s \approx \sqrt{m}$$

- Include those based on the eigenvectors corresponding to $\lambda_1(A_{G_p})$ and $\lambda_2(A_{G_p})$ [Arash Amini and Marvasti, 2015]
- A combinatorial construction overcomes this bottleneck with

$$s = \Omega(m^{\frac{1}{2}+\epsilon})$$

for small $\epsilon > 0$ [Bourgain et al., 2011b, Bourgain et al., 2011a]
- Accordingly, random constructions are abundant but deterministic constructions are hard to find ("hay in the haystack")

Paley matrices

- Matrices constructed from rows of the DFT matrix corresponding to QR's mod p [Bandeira et al., 2013] support

$$s \sim \sqrt{p}$$

- Conditioned on a conjecture about the # of edges in any subgraph of G_p [Bandeira et al., 2016], these matrices support

$$s \sim p/\text{polylog}(p)$$

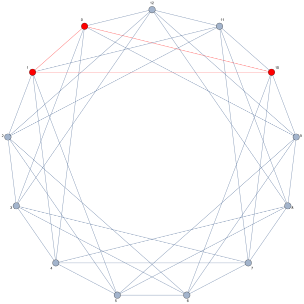
- Unconditional [Kaplan et al., 2019] for signals with a certain sparse structure

$$s = \Omega(m^{\frac{1}{2}+\frac{9}{40}})$$

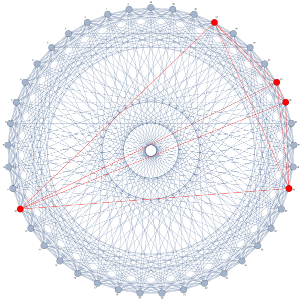
- A lower bound on $\omega(G_p)$ would lead to a lower bound on the distortion constant δ

Paley graph clique number

- Classic problem in number theory and additive combinatorics
- G_p share similarities with *Erdos-Renyi graphs* $\mathcal{G}(1/2, p)$
- Is $\omega(G_p) = O(\text{polylog } p)$, i.e., is G_p roughly a Ramsey graph?
- Note $\omega(\mathcal{G}(1/2, n)) \sim 2\log_2 n$



$\omega(G_{13}) = 3$



$\omega(G_{41}) = 5$

Existing bounds

- Upper bounds [Hanson and Petridis, 2021, Benedetto et al., 2021]
$$\omega(G_p) \leq (\sqrt{2p-1}+1)/2$$
- Improves on $\sqrt{p}$ by a constant prefactor.
- Lower bound for infinitely many primes [Graham and Ringrose, 1990]
$$\log p \cdot \log\log\log p \leq \omega(G_p)$$
- Conditioned on GRH [Montgomery, 1971],
$$\log p \cdot \log\log p \leq \omega(G_p)$$
- Numerical experiments [Bachoc et al., 2014]
$$\omega(G_p) \approx \mathrm{polylog}(p)$$

Integer program

- Easier to see in the context of the independence number $\alpha(G)$

$$\omega(G_p) = \max_{x \in \mathbb{R}^p} \sum_i x_i$$
$$\text{s.t. } x_i^2 = x_i \text{ for all } i \in V$$
$$x_i x_j = 0 \text{ for all } \{i, j\} \in E$$

- We focus on the clique problem $\omega(G)$ (i.e., take $x_i x_j = 0$ for all $\{i, j\} \notin E$)
- It makes connections to A_{G_p} more apparent

Nonconvex semidefinite matrix optimization

$$\max_{Y \in \mathbb{S}^{p+1 \times p+1}} \sum_{i \in \mathbb{Z}_p} Y_{\emptyset i}$$

$$\text{s.t. } Y_{ii}^2 = Y_{ii} \text{ for all } i \in V$$

$$Y_{ij} = 0 \text{ if } \{i, j\} \notin E$$

$$Y \succeq 0, \;\; Y_{\emptyset\emptyset} = 1$$

$$\text{rank}(Y) = 1$$

- This is equivalent to the previous program for $\omega(G_p)$
- Let $y = (1, x_1, \ldots, x_n)$ and reparametrize:

$$Y = yy^T = \begin{pmatrix} 1 & x_1 & x_2 & \ldots & x_p \\ x_1 & x_1 & x_1 x_2 & \ldots & x_1 x_p \\ x_2 & x_1 x_2 & x_2 & \ldots & x_1 x_p \\ \vdots & & & \ddots & \\ x_p & & & & x_p \end{pmatrix}$$

SOS-2 = Lovasz-Schrivjer$_0$ = L^1 convex relaxation

- Then we drop the nonconvex constraints

$$
\begin{aligned}
&\max \sum_{i \in V} y_i \\
&s.t. y \in \mathbb{R}^p, Y \in \mathbb{R}^{p \times p} \\
&Y_{ij} = 0 \text{ if } i \neq j, \{i,j\} \notin E \\
&Y_{ii} = y_i, \;\; i \in V \\
&\begin{pmatrix} 1 & y^\top \\ y & Y \end{pmatrix} \succeq 0
\end{aligned}
$$

- One can show this is equivalent to the Lovász ϑ function

SOS / Laserre-Parrilo hierarchy

- Denote the power sets of V of size $\leq t$ by $\mathcal{P}_t = \{S \subset V \mid |S| \leq t\}$.
- Now let $y = \left(\prod_{i \in S} x_i\right)_{S \in \mathcal{P}_t}$, $Y = yy^T$.
  - For example for $t = 2$, $y = (1, x_1, \ldots, x_n, x_1 x_2, \ldots x_2 x_1, \ldots x_p x_p)$.
- For $y \in \mathcal{P}_{2t}(V)$, $M_t(y)$ with $(M_t(y))_{I,J} = y_{I \cup J}$, $I, J \in \mathcal{P}_t(V)$ is called the moment matrix of y.

$$SOS_{2t}(G) = \max \sum_{i \in V} y_i$$
$$\text{s.t. } M_t(y) \succeq 0, y_0 = 1, y_{ij} = 0 \text{ if } \{i, j\} \in E.$$

Sum of squares relaxations

- An open problem proposed by Mixon and Bandeira is whether the SOS-4 relaxation of the Paley graph clique number breaks this barrier
- Xu & Kunisky
  - provided numerical evidence that $SOS_4(G_p)$ relaxation are $O(p^{\frac{1}{2}-\epsilon})$
  - proved an $\Omega(p^{\frac{1}{3}})$ lower bound
- However, $SOS_4(G_p)$ appears to be computationally intractable even for moderate $p \approx 250$.
- Gvozdenovic et al. introduced a more computationally efficient block-diagonal hierarchy of SDPs (L^t)

$$SOS_{2t}(G_p) \leq L^t(G_p)$$

Block Diagonal Hierarchy

- For $T \in \mathcal{P}_{t-1}(V)$, introduce $M(T;y)$, a principal sub-matrix of $M_t(y)$ indexed by $\bigcup_{S \subseteq T} \{S, S \cup \{i\}, i \in V\}$.

$$L^t(G) = \max \sum_{i \in V} y_i$$
$$M(T;y) \succeq 0 \ \forall \ |T| = t-1$$
$$y_0 = 1, y_{ij} = 0, \{i,j\} \in E.$$

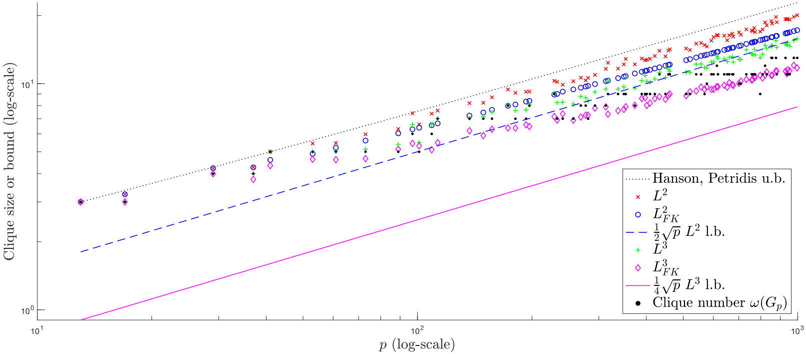
- Less computationally expensive than $SOS^{2t}(G)$.
- A relaxation of SOS because $M_t(y) \succeq 0$ is requires every submatrix to be PSD
- Block-diagonalized by zeta matrices - it is sufficient to use $p+1 \times p+1$ matrices in the constraints.

Main result: Lower bound on $L^t(G_p)$

- We proved the following lower bound

$$L^t(\overline{G}_p) \geq \frac{\sqrt{p}}{2^{t-1}} + o(\sqrt{p}).$$

- This shows L^t does not break $\sqrt{p}$ bottleneck for fixed t, but may beat it if $t(p)$ is a slowly increasing function of p.



Localization-relaxation trade-off

- Localization $G_{p,K}$ of subgraphs induced on vertices K adjacent to all vertices in G_p is another technique used to strengthen convex relaxations [Passuello, 2013, Magsino et al., 2019] and, more recently, spectral bounds on $\omega(G_p)$ [Kunisky, 2023].
- for any clique K of size a,

$$L^t(\overline{G}_{p,K}) \geq \frac{\sqrt{p}}{2^{a+t-1}} + o(\sqrt{p}). \tag{1}$$

- This shows L^t does not break $\sqrt{p}$ bottleneck for fixed t, but may beat it if $a(p)$ is a slowly increasing function of p.

Proof idea

- We construct a feasible point of L^2 using *Feige-Krauthgamer (FK) pseudomoments*, similarly to such construction in [Kunisky and Yu, 2022] for SOS_{2t}.
- The FK program $L^2_{FK}(G_p)$ corresponding to $L^2(G_p)$ is defined by replacing $A_{\{0\}}$ with:

$$A_{\{0\}} = \left(\begin{array}{c|c|c} y_{\{0\}} & y_{\{0\}} & y_{\{0,1\}}(A_{G_p})_{0,1:\text{end}} \\ \hline y_{\{0\}} & y_{\{0\}} & y_{\{0,1\}}(A_{G_p})_{0,1:\text{end}} \\ \hline \begin{array}{c} y_{\{0,1\}}\times \\ (A_{G_p})_{1:\text{end},0} \end{array} & \begin{array}{c} y_{\{0,1\}}\times \\ (A_{G_p})_{1:\text{end},0} \end{array} & y_{\{0,1\}}\text{diag}(A_{G_p})_{1:\text{end},0} + \alpha_3 M' \end{array}\right)$$

  where M' is the indicator matrix of triangles in G_p of the form $\{0, i, j\}$ for $1 \leq i, j < p$, and reducing the number of scalar optimization variables $y_{\{0,\alpha,\beta\}}$ corresponding to the orbits of triangles to the single $\alpha_3 \in \mathbb{R}$.
- Use the Schur complements to reduce the PSD constraints to a system of scalar inequalities for $y_{\{0\}}$ $y_{\{0,1\}}$ and α_3.

Future direction - symmetries and upper bounds

- We plan to upper bound L^2 and L^3, and therefore $\omega(G_p)$, by constructing feasible points of the corresponding dual programs.
- Since the edges and the edges triples (triangles) form orbits under $Aut(G_p)$, the number of optimization variables is proportional to the number of the representatives of such orbits
- Since a Paley graph is edge-transitive, the representatives of such orbits are given by $\{0, 1, \beta\}$ where both β and $\beta - 1$ are squares in $\mathbb{Z}_p$; there are approximately $(p-5)/24$ orbits.
- The upper bound problem can be reduced to a problem of studying the e.s.d of indicators of orbits as $p \to \infty$

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