PDE: HOMEWORK 11

Due Friday, December 2 (at the start of the recitation)

- (Evans, Problem 3.5.5(a).) Solve using characteristics:
$$x_1 u_{x_1} + x_2 u_{x_2} = 2u, \ u(x_1, 1) = g(x_1)$$
- Verify that $H(x-y)$ is a weak solution of
$$\begin{cases} u_x + u_y = 0, x \in \mathbb{R}, y \geqslant 0 \\ u(x,0) = H(x) \end{cases}$$
where H is the Heaviside function.

Date: November 29, 2016.

1

(Evans, Problem 3.5.5(a)): Solve using characteristics

$$\begin{cases} x_1 u_{x_1} + x_2 u_{x_2} = 2u \\ u(x_1, 1) = g(x_1) \end{cases}$$

<u>Soln:</u> By Eq (17), Evans §3.2.2 Example a.

$$F(\nabla u, u, x) = x_1 u_{x_1} + x_2 u_{x_2} - 2u = 0$$

$$F(\nabla u, u, x) = X \cdot \nabla u - 2u = 0$$

Therefore, we solve the following ODEs

$$\begin{cases} \dot{x}(s) = b(x(s)) = x(s) \\ \dot{z}(s) = -c(x(s))z(s) = 2z \end{cases}$$

$$\Rightarrow \begin{cases} \dot{x}_1 = x_1 \\ \dot{x}_2 = x_2 \\ \dot{z} = 2z \end{cases} \Rightarrow \begin{cases} x_1 = x_{01} e^s \\ x_2 = x_{02} e^s \\ z = z_0 e^{2s} \end{cases} \Rightarrow x_{02} = 1$$

since $\Gamma = (x_1 \in \mathbb{R}, x_2 = 1)$

$\Rightarrow$ to find the remaining constants, fix $(x_1, x_2) \in U$ and solve

$$(x_1, x_2) = e^s (x_{01}, 1) \Rightarrow s = \log x_2$$

$$x_{01} = \frac{x_1}{x_2}$$

$$u(x) = u(x_1, x_2) = z = g(x_{01}) e^{2s}$$

$$= g\left(\frac{x_1}{x_2}\right) e^{2\log x_2} = g\left(\frac{x_1}{x_2}\right) x_2^2$$

Verify that $H(x-y)$ is a weak solution of

$$\begin{cases} u_x + u_y = 0 & x \in \mathbb{R},\ y \geq 0 \\ u(x,0) = H(x) \end{cases}$$

where H is a Heavyside function

Soln Let $u = x+y$, $v = x-y$ $\Rightarrow$ $\varphi_x = \varphi_u \frac{\partial u}{\partial x} + \varphi_v \frac{\partial v}{\partial x} = \varphi_u + \varphi_v$

$\varphi_y = \varphi_u \frac{\partial u}{\partial y} + \varphi_v \frac{\partial v}{\partial y} = \varphi_u - \varphi_v$

$$J = \left| \det \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \right| = \left| \det \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \right| = 2$$

$$du\,dv = J\,dx\,dy = 2\,dx\,dy$$

The desired result follows if we show that

$$(*) \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(x-y)(\varphi_x + \varphi_y)\,dx\,dy + \int_{-\infty}^{\infty} H(x)\varphi\,dx \Big|_{y=0} = 0$$

for every $\varphi \in C_0^\infty(-\infty,\infty) \times (0,\infty)$

- note that $\int_{-\infty}^{\infty} H(x)\varphi(x,0)\,dx = 0$

  since the requirement of compact support on $(-\infty,\infty)\times(0,\infty)$ implies that $\varphi(x,0) = 0$ for all x.

- $4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(v)\varphi_u\,du\,dv = 4\int_0^\infty \varphi(u,v)\Big|_{u=-\infty}^{\infty} dv = 0$

  since φ is compactly supported.