PDE HW 1

1.1.3 (a) $(\alpha u_1+\beta u_2)_t-(\alpha u_1+\beta u_2)_{xx}+1=0$

$$\alpha(u_{1t}-u_{1xx})+\beta(u_{2t}-u_{2xx})=\underbrace{-1}_{\text{forcing term nonzero}}$$

$\Rightarrow$ 2nd order linear inhomogeneous

(b) $(\alpha u_1+\beta u_2)_t-(\alpha u_1+\beta u_2)_{xx}+x(\alpha u_1+\beta u_2)=0$

$$\alpha(u_{1t}-u_{1xx}+xu_1)+\beta(u_{2t}-u_{2xx}+xu_2)=0$$

$\Rightarrow$ 2nd order linear homogeneous

(c) $(\alpha u_1+\beta u_2)_t-(\alpha u_1+\beta u_2)_{xxt}+(\alpha u_1+\beta u_2)(\alpha u_1+\beta u_2)_x=0$

$$\alpha(u_{1t}-u_{1xxt})+\beta(u_{2t}-u_{2xxt})+\underbrace{\alpha^2u_1u_{1x}+\alpha u_1\beta u_{2x}+\alpha\beta u_{1x}u_2+\beta^2u_2u_{2x}}_{\neq\ \alpha(u_1u_{1x})+\beta(u_2u_{2x})}=0$$

$\Rightarrow$ 3rd order nonliner

(d) $(\alpha u_1+\beta u_2)_{tt}-(\alpha u_1-\beta u_2)_{xx}=\underbrace{x^2}$ $\leftarrow$ forcing term nonzero

$$=\alpha(u_{1tt}-u_{1xx})+\beta(u_{2tt}+u_{2xx})$$

$\Rightarrow$ 2nd order linear nonhomogeneous

(e) $i(\alpha u_1+\beta u_2)_t-(\alpha u_1+\beta u_2)_{xx}+\dfrac{\alpha u_1+\beta u_2}{x}=0$

$$=\alpha\left(iu_{1t}-u_{1xx}+\frac{u_1}{x}\right)+\beta\left(iu_{2t}-u_{2xx}+\frac{u_2}{x}\right)=0$$

$\Rightarrow$ 2nd order linear homogeneous

(f) $\dfrac{(2u)_x}{\sqrt{1+(2u_x)^2}}+\dfrac{(2u)_y}{\sqrt{1+(2u_y)^2}}=\dfrac{2u_x}{\sqrt{1+4u_x^2}}+\dfrac{2u_y}{\sqrt{1+4u_y^2}}\neq 2\left(\dfrac{u_x}{\sqrt{1+u_x^2}}+\dfrac{u_y}{\sqrt{1+u_y^2}}\right)$

$\Rightarrow$ 1st order nonlinear

(g) $(\alpha u_1+\beta u_2)_x+e^y(\alpha u_1+\beta u_2)_y=0$

$$\alpha(u_{1x}+e^yu_{1y})+\beta(u_{2x}+e^yu_{2y})$$

$\Rightarrow$ 1st order 1 linear homogeneous

(h) $u_t + u_{xxxx} = -\sqrt{1+u}$

$(u_t + u_{xxxx})^2 = 1+u \qquad 1+u>0$

$(u_t + u_{xxxx})^2 - u = 1$

$\Rightarrow$ 4th order nonlinear

1.1.4 $\alpha(u_1 - u_2) = \alpha u_1 - \alpha u_2 = g - g = 0$ (by linearity)

1.2.6 $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \Rightarrow y + C = \int \frac{dx}{\sqrt{1-x^2}}$ Let $x = \sin\theta$

$dx = \cos\theta\, d\theta$

$$y + C = \int \frac{\cos\theta\, d\theta}{\sqrt{1-\sin^2\theta}} = \theta = \sin^{-1}(x)$$

$y = \sin^{-1}(x) + C \leftarrow$ characteristic curves

$u(x,y) = f(y - \sin^{-1}(x)) = y - \sin^{-1}(x)$

by the fact that

$u(0,y) = y = f(y - \sin^{-1}(0)) =$

1.2.10

From 1.2.8
$$\begin{cases} x' = ax + by \\ y' = bx - ay \\ u_x = \frac{\partial u}{\partial x'} \cdot \frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial x} = a u_{x'} + b u_{y'} \\ u_y = \frac{\partial u}{\partial x'} \cdot \frac{\partial x'}{y} + \frac{\partial u}{\partial y'} \cdot \frac{\partial y'}{\partial y} = b u_{x'} - a u_{y'} \\ \ldots \text{ etc.} \end{cases}$$

we use this idea in 1.2.10

Let $x' = x + y$, $y' = x - y$ $\Rightarrow$ $u_x = u_{x'} \frac{\partial x'}{\partial x} + u_{y'} \frac{\partial y'}{\partial x} = u_{x'} + u_{y'}$

$u_y = u_{x'} \frac{\partial x'}{\partial y} + u_{y'} \cdot \frac{\partial y'}{\partial y} = u_{x'} - u_{y'}$

substituting the above into the original PDE

we get $u_{x'} + \cancel{u_{y'}} + u_{x'} - \cancel{u_{y'}} + u = 2u_{x'} + u = e^{y+2y}$

$$x+2y = \frac{3(x+y)-x-y}{2} = \frac{3x'-y'}{2}$$

$$2u_{x'}+u = e^{3x'-y'} \quad \text{use integrating factor } e^{\frac{x'}{2}}$$

$$e^{\frac{x'}{2}}u_{x'} + \frac{1}{2}e^{\frac{x'}{2}}u = \frac{1}{2}e^{\frac{4x'-y'}{2}}$$

$$\int \frac{\partial}{\partial x'}\left(e^{\frac{x'}{2}}u\right) = \int \frac{1}{2}e^{\frac{4x'-y'}{2}}\partial x'$$

$$e^{\frac{x'}{2}}u = \frac{1}{4}e^{\frac{4x'-y'}{2}} + f(y')$$

$$u(x',y') = \frac{1}{4}e^{\frac{3x'-y'}{2}}e^{-\frac{x'}{2}}f(y')$$

$$u(x,y) = \frac{1}{4}e^{x+2y} + e^{-\frac{(x+y)}{2}}f(x-y)$$

By the initial condition

$$\frac{1}{4}e^{x} + e^{-\frac{x}{2}}f(x) = 0 \Rightarrow f(x) = -\frac{1}{4}e^{\frac{3x}{2}}$$

$$\Longrightarrow u(x,y) = \frac{1}{4}e^{x+2y} - \frac{1}{4}e^{x-2y}$$

1.2.13 $u_x + 2u_y + (2x-y)u = 2x^2 + 3xy - 2y^2 = (2x-y)(x+2y)$

We look for a substitution that results in 1 independent variable.

$$\begin{aligned} x' &= x+2y \\ y' &= 2x-y \end{aligned} \Longrightarrow \begin{aligned} u_x &= u_{x'}\frac{\partial x'}{\partial x} + u_{y'}\cdot\frac{\partial y'}{\partial x} = u_{x'} + 2u_{y'} \\ u_y &= u_{y'}\frac{\partial y'}{\partial y} + u_{x'}\frac{\partial x'}{\partial y} = -u_{y'} + 2u_{x'} \end{aligned}$$

Thus from the original PDE we obtain

$5u_{x'} + y'u = x'y'$, which is a 1st order linear nonhomogeneous ODE

Using an integrating factor, we get

$$e^{\frac{x'y'}{5}}\left(u_{x'} + \frac{1}{5}y'u\right) = \frac{1}{5}e^{\frac{x'y'}{5}}x'y'$$

$$\frac{d}{dx_1}\left(e^{\frac{x'y'}{5}}u\right) = \frac{1}{5}e^{\frac{x'y'}{5}}x'y'$$

$$e^{\frac{x'y'}{5}}u = \int \frac{1}{5}e^{\frac{x'y'}{5}}x'y'$$

let $v = e^{\frac{x'y'}{5}}$
$dv = \frac{1}{5}e^{\frac{x'y'}{5}}y'$
and integrate by parts

$$= x'e^{\frac{x'y'}{5}} + \int e^{\frac{x'y'}{5}}x'\,dx'$$

$$= x'e^{\frac{x'y'}{5}} - \frac{5}{y'}e^{\frac{x'y'}{5}} + g(y')$$

Therefore $u = x' - \frac{5}{y'} + e^{-\frac{x'y'}{5}}g(y')$

$$= x + 2y - \frac{5}{2x-y} + e^{-\frac{(x+2y)(2x-y)}{5}} \cdot g(2x-y)$$