PDE: HOMEWORK 2

Due Friday, September 23 (at the start of the recitation)

- From the Strauss textbook: 1.5.4, 2.1.2, 2.1.5, 2.1.10, 2.2.2.
- Additional problem: Consider the following two PDEs:

$$\text{(Transport)} \qquad \begin{cases} u_t + u_x = 0 & t > 0, x \in \mathbb{R} \\ u(0,x) = f(x) & x \in \mathbb{R} \end{cases}$$

$$\text{(Diffusion)} \qquad \begin{cases} u_t - u_{xx} - u_{yy} = 0 & t > 0, (x,y) \in \mathbb{R}^2 \\ u(0,x,y) = g(x,y) & (x,y) \in \mathbb{R}^2 \end{cases}$$

(a) Show that the solution to the transport equation satisfies

$$\frac{\partial}{\partial t}\left(\int_{-\infty}^{\infty} u(t,x)^2 dx\right) = 0, \text{ for all } t > 0,$$

and hence

$$\int_{-\infty}^{\infty} u(t,x)^2 dx = \int_{-\infty}^{\infty} f(x)^2 dx, \text{ for all } t > 0.$$

Here you assume that all of these integrals are well-defined. In particular, assume that $u \to 0$ as $|x| \to \infty$. (Hint: try multiplying the transport equation by u and integrating in space.)

(b) Likewise, show that if u satisfies the diffusion equation above, then

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\iint_{\mathbb{R}^2} u(t,x,y)^2 dxdy\right) + \iint_{\mathbb{R}^2} |\nabla u|^2 dxdy = 0, \text{ for all } t > 0,$$

where $\nabla u = (u_x, u_y)$ and hence

$$\iint_{\mathbb{R}^2} u(t,x,y)^2 dxdy \leqslant \iint_{\mathbb{R}^2} g(x,y)^2 dxdy, \text{ for all } t > 0.$$

Again you may assume that all of these integrals are well-defined.

Date: September 21, 2016.

1.5.4 (a) it is easy to see that if $u(x,y,z)$ is a solution, then $u(x,y,z)+c$ where c is a constant is also a solution

(b) We have

$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \iiint_D \Delta u \,dx\,dy\,dz$$

$$= \iiint_D \nabla\cdot\nabla u \,dx\,dy\,dz = \iint_{\partial D} \nabla u\cdot n \,dS \quad (\text{Div Thm})$$

$$= \iint_{\partial D} \frac{du}{dn} = 0$$

(since $\nabla u \cdot n$ is the directional derivative of u in the direction n)

(c) In the case of heat, for example, part (a) means that if a distribution of heat satisfies this equation, then increasing or decreasing this distribution by a constant at all points of D will also satisfy this equation.
Part (b) means there is no net heat transferred though ∂D (the boundary of D)

2.1.2 By the d'Alembert formula

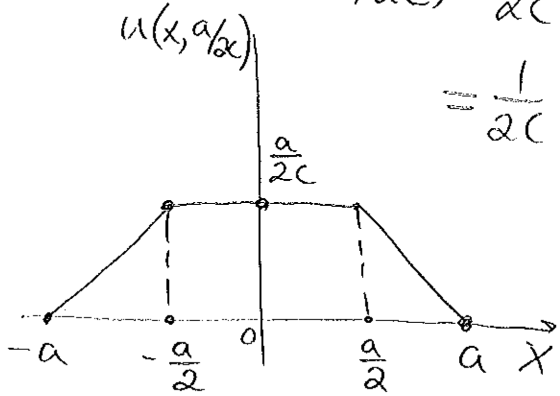
$$u(x,t) = \frac{1}{2}\left[\log(1+(x+ct)^2) + \log(1+(x-ct)^2)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} 4+s\,ds$$

$$= \frac{1}{2}\log\left[(1+(x+ct)^2)(1+(x-ct)^2)\right] + \frac{1}{2c}\left[4s+\frac{s^2}{2}\right]_{s=x-ct}^{x+ct}$$

$$= \frac{1}{2}\log\left[(1+(x+ct)^2)(1+(x-ct)^2)\right] + \frac{1}{2c}\left[8ct + \frac{(x+ct)^2-(x-ct)^2}{2}\right]$$

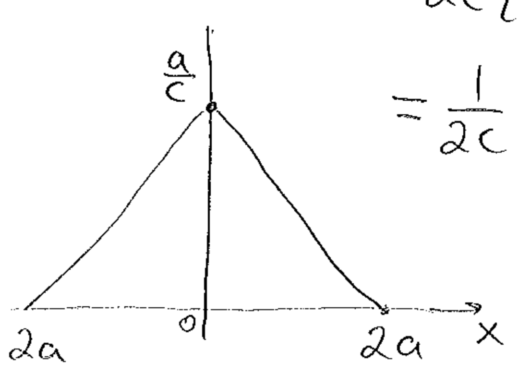
$$= \frac{1}{2}\log\left[(1+(x+ct)^2)(1+(x-ct)^2)\right] + 4t + tx$$

2.1.5 $u(x, a/2c) = \frac{1}{2c}\left\{\text{length}\left(x-\frac{a}{2}, x+\frac{a}{2}\right) \cap (-a,a)\right\}$



$$= \frac{1}{2c}\begin{cases} a & \text{if } |x| < \frac{a}{2} \\ a-x & \text{if } \frac{a}{2} < x < a \\ x-a & \text{if } -a < x < -\frac{a}{2} \\ 0 & \text{if } |x| > a \end{cases}$$

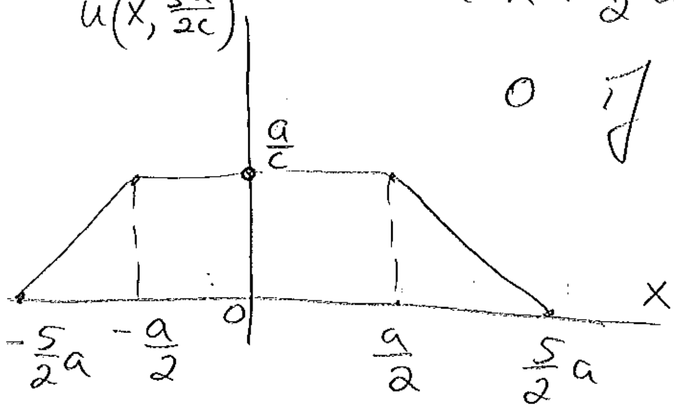
$u(x, \frac{a}{c}) = \frac{1}{2c}\left\{\text{length}(x-a, x+a) \cap (-a,a)\right\}$



$$= \frac{1}{2c}\begin{cases} 2a-x & \text{if } x \in (0, 2a] \\ 0 & \text{if } |x| > a \\ 2x+a & \text{if } x \in [-2a, 0] \end{cases}$$

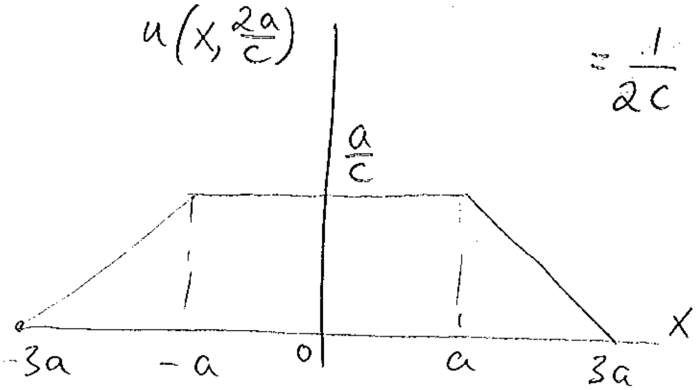
$$u\left(x, \frac{3a}{2c}\right) = \frac{1}{2c}\left\{ \text{length}\left(x - \frac{3a}{2}, x + \frac{3a}{2}\right) \cap (-a, a) \right\}$$

$$= \frac{1}{2c} \begin{cases} 2a & \text{if } |x| < \frac{a}{2} \\ \frac{5}{2}a - x & \text{if } x \in \left[\frac{a}{2}, \frac{5}{2}a\right] \\ x + \frac{5}{2}a & \text{if } x \in \left[-\frac{5}{2}, -\frac{a}{2}\right] \\ 0 & \text{if } |x| > \frac{5}{2} \end{cases}$$



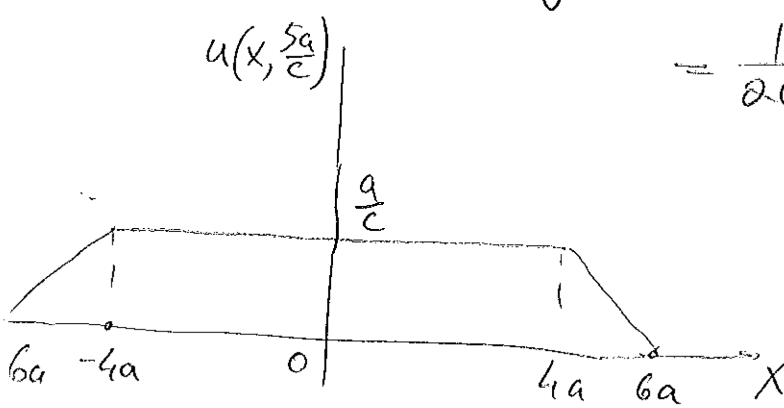
$$u\left(x, \frac{2a}{c}\right) = \frac{1}{2c}\left\{ \text{length}\left(x - 2a, x + 2a\right) \cap (-a, a) \right\}$$

$$= \frac{1}{2c} \begin{cases} 2a & \text{if } |x| < a \\ 3a - x & \text{if } x \in [a, 3a] \\ x - 3a & \text{if } x \in [-3a, -a] \\ 0 & \text{if } |x| > 3a \end{cases}$$



$$u\left(x, \frac{5a}{c}\right) = \frac{1}{2c}\left\{ \text{length}\left(x - 5a, x + 5a\right) \cap (-a, a) \right\}$$

$$= \frac{1}{2c} \begin{cases} 2a & \text{if } |x| < 4a \\ 6a - x & \text{if } x \in [4a, 6a] \\ x + 6a & \text{if } x \in [-6a, -4a] \\ 0 & \text{if } |x| > 6a \end{cases}$$



2.1.10 By factoring the given operator

$$\left(5\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)\left(-4\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)u=0$$

By introducing a new function v, we get the following linear system

$$v=-4u_t+u_x \quad (*)$$

$$5v_t+v_x=0 \quad (**)$$

By Section 1.2 of the Strauss textbook $(**)$ has the solution $v=h(5x-t)$

Then we solve

$$-4u_t+u_x=h(5x-t)$$

We can see that $u(x,t)=f(5x-t)$ is a particular solution:

$$-4u_t+u_x=4f'(5x-t)+5f'(5x-t)$$

is a function of $5x-t$, so that $f'=\frac{h}{9}$

The solution to the homogeneous equation is given by $g(4x+t)$

Therefore, the most general solution to the original equation is given by summing the general and the particular solution

$$u(x,t)=f(5x-t)+g(4x+t)$$

2.2.2 (a) Assuming that the mixed partials are smooth

$$\frac{\partial e}{\partial t} = u_t \cdot u_{tt} + u_x \cdot u_{xt}$$
$$\frac{\partial p}{\partial x} = u_{tx} u_x + u_t u_{xx} = u_{xt} u_x + u_t u_{tt} \implies \frac{\partial e}{\partial t} = \frac{\partial p}{\partial x}$$

Similarly,

$$\frac{\partial p}{\partial t} = u_{tt} u_x + u_t u_{xt}$$
$$\frac{\partial e}{\partial x} = u_t u_{tx} + u_x u_{xx} = u_t u_{xt} + u_x u_{tt} \implies \frac{\partial e}{\partial x} = \frac{\partial p}{\partial t}$$

(b) $$e_{tt} = u_{tt} \cdot u_{tt} + u_t \cdot u_{ttt} + u_{xt} \cdot u_{xt} + u_x \cdot u_{xtt}$$

$$e_{xx} = u_{tx} u_{tx} + u_t u_{txx} + u_{xx} u_{xx} + u_x u_{xxx}$$
$$= u_{xt} u_{xt} + u_t u_{ttt} + u_{tt} u_{tt} + u_x u_{xtt}$$

Similarly

$$p_{xx} = u_{txx} u_x + u_{tx} u_{xx} + u_{tx} u_{xx} + u_t u_{xxx}$$
$$p_{tt} = u_{ttt} u_x + u_{tt} u_{xt} + u_{tt} u_{xt} + u_t u_{xtt} \implies p_{xx} = p_{tt}$$
$$= u_{txx} u_x + u_{tx} u_{xx} + u_{tx} u_{xx} + u_t u_{xxx}$$

Additional Problem:

(a) We multiply the transport equation by u and integrate in space

$$0 = \int_{-\infty}^{\infty} u(u_t + u_x)\,dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial t}\left[\frac{u^2}{2}\right] dx + \int_{-\infty}^{\infty} u\, u_x\, dx$$
$$= \underbrace{\frac{\partial}{\partial t} \int_{-\infty}^{\infty} \frac{u^2}{2}\,dx}_{(*)} + \underbrace{\left[\frac{u^2}{2}\right]_{x=-\infty}^{\infty}}_{\text{vanishes}}$$

$$\implies (*) = 0 \implies_{(**)} \int_{-\infty}^{\infty} u(t,x)^2\,dx = h(x) \quad \text{(i.e. constant w/r/t to } t\text{)}$$

(6)

Therefore

$$\int_{-\infty}^{\infty} u(t,x)^2 dx = h(x) = \int_{-\infty}^{\infty} u(0,x)^2 dx = \int_{-\infty}^{\infty} f(x)^2 dx$$

(since $h(x)$ is again constant w/r/t

(6) We multiply the diffusion equation by u and integrate over $\mathbb{R}^2$

$$\iint_{\mathbb{R}^2} u u_t - u(u_{xx} + u_{yy})\, dx\, dy$$

$$= \frac{\partial}{\partial t} \iint_{\mathbb{R}^2} \frac{u^2}{2} dx dy - \underbrace{\left[\iint_{\mathbb{R}^2} u u_{xx} dx dy + \iint_{\mathbb{R}^2} u u_{yy} dy dx \right]}_{(*)}$$

Integrating by parts we get

$$\int_{-\infty}^{\infty} u u_{xx} dx = u u_x \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u_x \cdot u_x \, dx = -\int_{-\infty}^{\infty} u_x^2 \, dx$$

vanishes by hypothesis on u

similarly: $\displaystyle\int_{-\infty}^{\infty} u u_{yy} dy = -\int_{-\infty}^{\infty} (u_y)^2 dy$

Therefore $\displaystyle (*) = \iint_{\mathbb{R}^2} u_x^2 + u_y^2 \, dxy$

$$= \iint_{R^2} \nabla u \cdot \nabla u \, dx dy = \iint_{\mathbb{R}^2} |\nabla u|^2 dx dy$$