PDE: HOMEWORK 3

Due Friday, September 30 (at the start of the recitation)

- From the Strauss textbook: 2.3.4, 2.3.8, 2.4.4, 2.4.15, 2.4.16.
- Additional problem: Suppose that $u = u(t,x)$ is a positive function (i.e. $u > 0$), and u solves the equation

  $$(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2})u + u = 0, \quad \text{for } (t,x) \in \Omega_T := (0,T) \times [-L,L]$$

  Show that u satisfies the maximum principle.

Date: September 24, 2016.

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2.4.15, 2.4.16.
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$\times [-L, L]$

2.3.4.

(a) Since u equals zero on the lateral sides and the maximum value at time $t = 0$ is $u(1/2, 0) = 1$, the strong maximum principle implies that $u(x,t) < 1$ for all $t > 0$ and $0 < x < 1$. Since the minimum value at time $t = 0$ is $u(0,0) = 0$, the strong minimum principle implies that $u(x,t) > 0$ for all $t > 0$ and $0 < x < 1$.

(b) Let $v(x,t) = u(1-x,t)$. Then $v_x(x,t) = -u_x(1-x,t)$, so

$$v_{xx}(x,t) = u_{xx}(1-x,t) = u_t(1-x,t) = v_t(x,t),$$

and thus v is also a solution of the wave equation. Since $v(0,t) = u(1,t) = 0$, $v(1,t) = u(0,t) = 0$ and $v(x,0) = u(1-x,0) = 4(1-x)(1-(1-x)) = 4x(1-x)$, v is a solution of the diffusion equation with the same initial data and boundary data as u. Since solutions are unique, it follows that $v(x,t) = u(x,t)$ for all $t \geq 0$ and $0 \leq x \leq 1$.

(c) Since u vanishes as $x = 0$ and $x = 1$ for all t, we have

$$\begin{aligned}
\frac{d}{dt}\int_0^1 u^2(x,t)\,dx &= 2\int_0^1 u(x,t)u_t(x,t)\,dx \\
&= 2\int_0^1 u(x,t)u_{xx}(x,t)\,dx \\
&= 2u(x,t)u_x(x,t)\Big|_{x=0}^{x=1} - 2\int_0^1 u_x^2(x,t)\,dx \\
&= -2\int_0^1 u_x^2(x,t)\,dx.
\end{aligned}$$

The integral cannot be zero since this would imply $u_x(x,t) = 0$, which means that for each t, $u(x,t)$ would be constant in x. Since $u(0,t) = 0$ that constant would be zero. But by part (a), u is positive for $0 < x < 1$. Thus $\int_0^1 u^2(x,t)\,dx$ is strictly decreasing, since its derivative is negative.

2.3.8

$$\frac{d}{dt}\frac{1}{2}\int_0^l u^2\,dx = \int_0^l u u_t\,dx = k\int_0^l u_{xx}u\,dx \quad \text{(integrate by parts)}$$

$$= u u_x \Big|_{x=0}^{l} - \int_0^l (u_x)^2\,dx \qquad \begin{array}{l} u=u \\ dv = u_{xx} \\ v = u_x \\ du = u_x\,dx \end{array}$$

$$= u(l,t)u_x(l,t) - u(0,t)u_x(0,t) - \int_0^l (u_x)^2\,dx$$

$$= -a_l u^2(l,t) - a_0 u^2(0,t) - \int_0^l (u_x)^2\,dx \qquad \begin{bmatrix} u_x(0,t) = a_0 u(0,t) \\ u_x(l,t) = -a_l u(l,t) \end{bmatrix}$$

≤ 0 by the fact that $a_0, a_l > 0$

2.4.4 By Eq (2.4.8)

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} \exp(-(x-y)^2/4kt)\,\phi(y)\,dy$$

$$= \frac{1}{\sqrt{4\pi kt}}\int_0^{\infty} \exp(-(x-y)^2/4kt)\,e^{-y}\,dy \quad (*)$$

We complete the square.

$$(**)\ (x-y)^2 - y^2 = (y^2 + 4kty - 2xy - 4ktx + 4k^2t^2 + x^2) - 4ktx + 4k^2t^2$$

$$\frac{(**)}{4kt} = \frac{-(y^2+2kt-x)^2}{4kt} + kt - x$$

and let $z = \frac{y+2kt-x}{\sqrt{4kt}}$

$$dz = \frac{1}{\sqrt{4kt}}\,dy \Rightarrow u(x,t) = \frac{1}{\sqrt{\pi}}e^{kt-x}\int_{\frac{2kt-x}{\sqrt{4kt}}}^{\infty} e^{-z^2}\,dz$$

2.4.15

Let u_1 and u_2 be distinct solutions to the Neumann boundary problem; let $v = u_1 - u_2$

Then $v_t - k v_{xx} = 0$

$$0 = \int_0^\ell v\,(v_t - k v_{xx})\,dx$$

$$= \frac{1}{2}\int_0^\ell \frac{\partial}{\partial t}(v^2)\,dx - k\int_0^\ell v v_{xx}\,dx$$

$$= \frac{1}{2}\int_0^\ell \frac{\partial}{\partial t}(v^2)\,dx - k\int_0^\ell \left(\frac{\partial}{\partial x}(v v_x) - v_x^2\right) dx$$

$$= \frac{1}{2}\frac{\partial}{\partial t}\int_0^\ell v^2\,dx \underbrace{- k v v_x \Big|_{x=0}^{\ell}} + \int_0^\ell v_x^2\,dx$$

vanishes since by construction of v: $v_x(0,t) = g(t) - g(t) = 0$

$v_x(\ell,t) = h(t) - h(t) = 0$

Since the last term is non negative, we have $\frac{\partial}{\partial t}\int_0^\ell v^2\,dx \leq 0$. Therefore, $\int_0^\ell v^2\,dx$ is nonincreasing as a function of t, i.e

$$\int_0^\ell v^2(x,t)\,dx \leq \int_0^\ell v^2(x,0)\,dx = 0$$

since again by construction of v $v(x,0) = \varphi(x) - \varphi(x) = 0$

Since $\int_0^\ell v^2(x,t)\,dx$ is nonnegative we have $v \equiv 0$ on $(0,\ell)$ as desired

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2.4.16 Let $u(x,t) = e^{-bt} v(x,t)$. Then the original equation becomes

$$e^{-bt}\left(-\cancel{bv} + v_t - k v_{xx} + \cancel{bv}\right) = 0$$

This implies that

$$\begin{cases} v_t = k v_{xx} \\ v(x,0) = \varphi(x) \end{cases}$$

By Eqn (2.4.8) in the ~~text book~~

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \phi(y)\, dy$$

$\Rightarrow u(x,t) = e^{-bt} v(x,t)$ as above

Add'l Problem If u assumes a maximum at an interior point (x,t), then $u_{xx}(x,t) \le 0$ and $u_t(x,t) = 0$ but by the PDE we have

$$u_t(x,t) = u_{xx}(x,t) - u(x,t) < 0,$$ which is a contradiction