PDE: HOMEWORK 4

Due Friday, October 7th (at the start of the recitation)

- From the Strauss textbook: 4.1.4, 4.1.6, 4.2.2, 4.2.4.
- Additional problem: Consider the heat equation $u_t = u_{xx}$ on the bounded domain $0 \leqslant x \leqslant 1$ with boundary data $u(0,t) = 0$ and $u(1,t) = 1$. Find a solution satisfying the initial condition $u(x,0) = x$.

Date: October 2, 2016.

PDE HW 4

4.1.4 We look for a solution in the form $u(x,t) = X(x)T(t)$. From the original ODE we have

$$XT'' = c^2 X''T - rXT'$$

$$-\frac{T''}{c^2T} = -\frac{X''}{X} + \frac{rT'}{T} \qquad \text{(divide by } -c^2XT\text{)}$$

$$-\frac{T''+rT'}{c^2T} = -\frac{X''}{X} = \lambda = \beta^2 \quad , \beta > 0$$

By the same argument as in Section 4.1 of the textbook $\lambda > 0$ and

$$\lambda_n = \left(\frac{n\pi}{\ell}\right)^2 \text{ and } X_n(x) = \sin\frac{n\pi x}{\ell} \quad (n = 1, 2, 3 \ldots)$$

Next observe that by ODE Theory to solve

$$T'' + rT' + c^2\left(\frac{n\pi}{\ell}\right)^2 T = 0$$

we consider the roots of the characteristic polynomial

$$y^2 + ry + c^2\left(\frac{n\pi}{\ell}\right)^2 = 0$$

$$y = \frac{-r \pm \sqrt{r^2 - 4\left(\frac{cn\pi}{\ell}\right)^2}}{2}$$ which are complex-valued since $0 < r < 2\pi c/\ell$

Therefore

$$T_n(t) = A_n e^{-\frac{rt}{2}} \cos\left(\frac{\sqrt{4\left(\frac{cn\pi}{\ell}\right)^2 - r^2}}{2}t\right) + B_n e^{-\frac{rt}{2}} \sin\left(\frac{\sqrt{4\left(\frac{cn\pi}{\ell}\right)^2 - r^2}}{2}t\right)$$

$T_n(t)$

consequently

$$u(x,t) = \sum_n \left(A_n e^{\alpha t}\cos(\beta_n t) + B_n e^{\alpha t}\sin(\beta_n t)\right)\sin\frac{n\pi x}{\ell}$$

is a solution where $\alpha = -\frac{r}{2}$, $\beta_n = \frac{\sqrt{4\left(\frac{cn\pi}{\ell}\right)^2 - r^2}}{2}$

observe that

$$\psi(x) = u_t(x,0) = X(x)\,T'(0)$$

$$T'(t) = \alpha e^{\alpha t}\left(A_n\cos\beta_n t + B_n\sin(\beta_n t)\right) + e^{\alpha t}\left(-\beta_n A_n\sin\beta_n t + \beta_n B_n\cos(\beta_n t)\right)$$

$$T'(0) = \alpha A_n + \beta_n B_n$$

Therefore, the intial conditions must be of the form

$$\varphi(x) = \sum A_n \sin\frac{n\pi x}{\ell}$$

$$\psi(x) = (\alpha A_n + \beta_n B_n)\sin\frac{n\pi x}{\ell}$$

4.1.6 As in the preceding problem, we look for a solution in the form

$$u(x,t) = X(x)T(t).$$

From the original ODE we have

$$tXT' = X''T + 2XT \quad (\text{divide by } -XT)$$

$$-\frac{tT'}{T} + 2 = -\frac{X''}{X} = \lambda$$

Again by the same argument as in Section 4.1 of the textbook $\lambda > 0$ and

$$\lambda_n = n^2 \quad \text{and} \quad X_n(x) = \sin(nx) \; (n=1,2 \dots)$$

Next we solve the ODE

$$tT' + (n^2-2)T = 0 \quad (\text{divide by } tT)$$

$$\int \frac{T'}{T} dt = \int \frac{2-n^2}{t} dt$$

$$\log T = (2-n^2)\log t + C_n'$$

$$T = C_n t^{2-n^2}$$

Thus $u(x,t) = \sum_n C_n t^{2-n^2} \sin(nx)$ is a general solution

$$\Rightarrow u(x,0) = \sum_n C_n \cdot 0 \sin(nx) = 0 \text{ for any } C_n$$

Therefore we have infinitely many solutions

4.2.2 And again we look for a solution in the form $u(x,t) = X(x)T(t)$

As in (4.1.6) of the text book, for $\lambda = \beta^2 > 0$

$$X(x) = C\cos\beta x + D\sin\beta x$$
$$X'(x) = -C\beta\sin\beta x + D\beta\cos\beta x$$

The Neumann boundary condition on the left means that $0 = X'(0) = D\beta$, so $D = 0$

The Dirichlet boundary condition on the right means that

$0 = X(\ell) = C\cos\beta\ell$

which implies that $\beta = \frac{(n+\frac{1}{2})\pi}{\ell}$ (since we don't want $C = 0$)

To check whether zero is an eigenvalue set $\lambda = 0$ in $-X'' = 0$ so that $X(t) = C + Dx$ and $X'(t) = D$. By the Neuman boundary $D = 0$ and by the Dirichlet boundary $C = 0$, which implies that zero is not an eigenvalue

To check whether there could be negative eigenvalues, we let $\lambda = -\gamma^2$ and $X'' = \gamma^2 X$ so by ODE theory $X(x) = C\cosh\gamma x + D\sinh\gamma x$ and $0 = X(\ell) = C$ $(\cosh 0 = 1)$

Thus $\beta_n^2 = \lambda_n = \left(\frac{(n+\frac{1}{2})\pi}{\ell}\right)^2$ and $X_n = \cos\beta_n x$

$X'(x) = D\cosh x$ and $0 = X'(0) = D$

since $\cosh 0 = 1$ again

Therefore the eigenvalues are non-negative

Lastly, to check whether λ can be complex let γ be one of the square roots of $-\lambda$ and the other one is $-\gamma$.

Then $X(x) = Ce^{\gamma x} + De^{-\gamma x}$ (where we have a complex exponential function)

and $X'(x) = C\gamma e^{\gamma x} + \gamma D e^{-\gamma x}$

Therefore $0 = X'(0) = C\gamma + D\gamma \Rightarrow C + D = 0$

and $0 = X(\ell) = Ce^{\gamma \ell} + De^{-\gamma \ell}$

Therefore, as discussed in §4.1.1 $e^{2\gamma \ell} = 1$ which implies that $Re(\gamma) = 0$ and

$2\ell\, Im(\gamma) = 2\pi n$ for some integer n.

Hence $\gamma = n\pi i/\ell$ and $\lambda = -\gamma^2 = \pi^2 n^2/\ell$

which is real positive

As discussed in §4.1.1 of the textbook

$T_n = A_n \cos\beta_n ct + B_n \sin\beta_n ct$

and therefore

$$u(x,t) = \sum_n (A_n \cos\beta_n ct + B_n \sin\beta_n ct)\cos\beta_n x$$

where $\beta_n = \frac{(n+\frac{1}{2})\pi}{\ell}$

4.2.4

(a) Separation of variables leads to the eigenvalue problem

$$\begin{cases} X'' = -\lambda X & 0 < x < l \\ X(-l) = X(l) \\ X'(-l) = X'(l). \end{cases}$$

Looking for positive eigenvalues $\lambda = \beta^2 > 0$ implies

$$X(x) = C\cos(\beta x) + D\sin(\beta x).$$

The boundary condition $X(-l) = X(l)$ implies $D\sin(\beta l) = 0$, so $\beta = \frac{n\pi}{l}$. The boundary condition $X'(-l) = X'(l)$ implies $C\beta\sin(\beta l) = 0$, so $\beta = \frac{n\pi}{l}$. Therefore,

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2 \qquad X_n(x) = C_n\cos\left(\frac{n\pi x}{l}\right) + D_n\sin\left(\frac{n\pi x}{l}\right)$$

for $n = 1, 2, 3, \ldots$

If $\lambda = 0$, then $X(x) = C + Dx$. The boundary condition $X(-l) = X(l)$ implies $D = 0$. The boundary condition $X'(-l) = X'(l) = 0$ will be satisfied for arbitrary C. Therefore, $\lambda = 0$ is an eigenvalue with corresponding eigenfunction $X(x) = C$.

If $\lambda = -\gamma^2 < 0$, then $X(x) = C\cosh(\gamma x) + D\sinh(\gamma x)$. The boundary condition $X(-l) = X(l)$ implies $D\sinh(\gamma l) = 0$, so $D = 0$. The boundary condition $X'(-l) = X'(l)$ implies $C\gamma\sinh(\gamma l) = 0$, so $C = 0$. Therefore, all the eigenvalues are $\lambda_n = (n\pi/l)^2$ for $n = 0, 1, 2, 3, \ldots$

(b) Solving the equation $T_n' = -\lambda_n k T_n$ gives $T_n(t) = A_n e^{-k\lambda_n t}$. Therefore,

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n\cos\left(\frac{n\pi x}{l}\right) + B_n\sin\left(\frac{n\pi x}{l}\right)\right)e^{-kn^2\pi^2 t/l^2}.$$

Additional problem: It is straightforward to see that the stationary solution
$u(x,t) = x$ satisfies this IVP
(separation of variables would result in $\lambda = 0$, which would also suggest a linear solution)