PDE: HOMEWORK 5

Due Friday, October 14th (at the start of the recitation)

- From the Strauss textbook: 5.1.5, 5.1.9, 5.2.10, 5.2.11, 5.3.6
- Additional problems
  - Suppose that $u = u(x,t)$ solves the equation
    $$u_t - u_{xx} = -1 - x^2$$
    for $(x,t) \in \Omega_T := [-L, L] \times (0,T)$. Show that u cannot attain its maximum value at a point in the interior of Ω_T.
  - By factoring the operator, find an explicit solution to the problem
    $$u_{tt} + u_{tx} - 2u_{xx} = 0, \ t \geqslant 0, x \in \mathbb{R}$$
    with initial data
    $$u(x,0) = 0; \ u_t(x,0) = x$$

Date: October 8, 2016.

5.1.5.

(a) The sine series for $\phi(x) = x$ is

$$x = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}2l}{m\pi} \sin \frac{m\pi x}{l}$$

by equation (5.1.12). Integrating term-by-term gives

$$\frac{1}{2}x^2 = C + \sum_{m=1}^{\infty} \frac{(-1)^m 2l^2}{m^2\pi^2} \cos \frac{m\pi x}{l}.$$

The constant C must be

$$C = \frac{1}{2}A_0 = \frac{1}{l}\int_0^l \frac{1}{2}x^2\,dx = \frac{l^2}{6},$$

so

$$\frac{1}{2}x^2 = \frac{l^2}{6} + \sum_{m=1}^{\infty} \frac{(-1)^m 2l^2}{m^2\pi^2} \cos \frac{m\pi x}{l}.$$

(b) Evaluating at $x = 0$ yields

$$\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m^2} = \frac{\pi^2}{12}.$$

5.1.9. Since $\phi(x) = 0$ and $\psi(x) = \cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$, the coefficients in the cosine series for ϕ are $A_n = 0$ for all n, and the coefficients in the cosine series for ψ are $B_0 = 1$, $B_2 = \frac{1}{4c}$ and $B_n = 0$ for all other n. Hence equation (4.2.7) implies

$$u(x,t) = \frac{1}{2}t + \frac{1}{4c}\sin 2ct \cos 2x.$$

5.2.10.

(a) By definition $\phi_{\text{odd}}(x) = -\phi(-x)$ for $-l < x < 0$, and $\phi_{\text{odd}}(x) = \phi(x)$ for $0 < x < l$. So in order for ϕ_{odd} to be continuous at $x = 0$ we need

$$\lim_{x \to 0^+} \phi(x) = -\lim_{x \to 0^-} \phi(-x).$$

The limit on the right is the same as $-\lim_{x \to 0^+} \phi(x)$, so we must have $\lim_{x \to 0^+} \phi(x) = 0$.

(b) By part (a) we require

$$\lim_{x \to 0^+} \phi(x) = \lim_{x \to l^-} \phi(x) = 0$$

in order for ϕ_{odd} to be continuous. In order for ϕ_{odd} to also be differentiable at $x = 0$ we need

$$\phi'_{\text{odd}}(0) = \lim_{h \to 0} \frac{\phi_{\text{odd}}(h) - \phi_{\text{odd}}(0)}{h} = \lim_{h \to 0} \frac{\phi_{\text{odd}}(h)}{h}$$

to exist. Evaluating this from the right and using L'Hopital's rule, we have

$$\lim_{h \to 0^+} \frac{\phi_{\text{odd}}(h)}{h} = \lim_{h \to 0^+} \frac{\phi(h)}{h} = \lim_{h \to 0^+} \phi'(h).$$

On the other hand, the limit from the left is

$$\lim_{k\to 0^-} \frac{\phi_{\text{odd}}(k)}{k} = \lim_{k\to 0^-} \frac{-\phi(-k)}{k} = \lim_{k\to 0^-} \phi'(-k) = \lim_{h\to 0^+} \phi'(h).$$

Since the left and right hand limits agree, $\phi'_{\text{odd}}(0)$ exists provided the limit $\lim_{h\to 0^+} \phi'(h)$ exists.

(The result of part (b) is sometimes summarized informally by saying that ϕ is continuously differentiable on $[0, l]$ and $\phi(0) = 0$.)

5.2.11. The complex form of the Fourier series for e^x is $\sum_{n=-\infty}^{\infty} C_n e^{in\pi x/l}$, where

$$\begin{aligned} C_n &= \frac{1}{2l} \int_{-l}^{l} e^x e^{-in\pi x/l}\,dx = \left.\frac{e^{x(1-in\pi/l)}}{2l(1-in\pi/l)}\right|_{-l}^{l} \\ &= \frac{e^{l-in\pi} - e^{-l+in\pi}}{2l(1-in\pi/l)} = \frac{(-1)^n \sinh(l)}{l(1-in\pi/l)} \\ &= \frac{(-1)^n (l+in\pi)\sinh(l)}{l^2+n^2\pi^2}. \end{aligned}$$

The real form is therefore

$$\frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x/l) + B_n \sin(n\pi x/l)$$

where

$$A_n = C_n + C_{-n} = \frac{2l(-1)^n \sinh(l)}{l^2+n^2\pi^2}$$

and

$$B_n = i(C_n - C_{-n}) = \frac{(-1)^{n+1} 2n\pi \sinh(l)}{l^2+n^2\pi^2}.$$

5.3.6. The solution of $X' = \lambda X$ is $X = Ce^{\lambda x}$. The boundary condition $X(0) = X(1)$ therefore implies $e^{\lambda} = 1$, so $\lambda = 2n\pi i$. The eigenfunctions are $X_n = e^{2\pi i n x}$ and since

$$(X_m, X_n) = \int_0^1 X_m(x)\overline{X_n(x)}\,dx = \int_0^1 e^{2\pi i x(m-n)}\,dx = \left[\frac{e^{2\pi i x(m-n)}}{2\pi i(m-n)}\right]_0^1 = 0$$

for $m \neq n$, they are orthogonal.

Add' problems

- If u attains its maximum at an interior point (x_0, y_0) then $u_t(x_0, y_0) = 0$ and $u_{xx} \leq 0$. This implies that $-1 - x^2 > 0$ which is a contradiction
- We factor the operator and change coordinates such that the PDE is reduced to a single mixed partial derivative term

$$(*) \quad u_{tt} + u_{tx} - 2u_{xx} = \left(\frac{\partial}{\partial t} + 2\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)u = 0$$

Let $\eta = x - 2t$ and $\nu = x + t$

$$u_t = u_\eta \frac{\partial \eta}{\partial t} + u_\nu \frac{\partial \nu}{\partial t} = -2u_\eta + u_\nu$$

$$u_x = u_\eta \frac{\partial \eta}{\partial x} + u_\nu \frac{\partial \nu}{\partial t} = u_\eta + u_\nu$$

$$\Rightarrow \begin{cases} \frac{\partial}{\partial t} = -2\frac{\partial}{\partial \eta} + \frac{\partial}{\partial \nu} \\ \frac{\partial}{\partial x} = \frac{\partial}{\partial \eta} + \frac{\partial}{\partial \nu} \end{cases}$$

$$(*) = \left(3 \cdot \frac{\partial}{\partial \nu}\right)\left(-3\frac{\partial}{\partial \eta}\right)u = 0 \Rightarrow u_{\nu\eta} = 0$$

$$u(\eta, \nu) = h(\eta) + g(\nu)$$

Therefore $u(x,y) = h(x-2t) + g(x+t)$

$u(x,0) = 0 \Rightarrow h(x) + g(x) = 0 \Rightarrow g = -h$

Therefore

$$u(x,t) = -g(x-2t) + g(x+t)$$

$$u_t(x,t) = 2g_t(x-2t) + g_t(x+t)$$

$$u_t(x,0) = 2g_t(x) + g_t(x) = x$$

$$g_t(x,0) = \frac{x}{3}$$

$$g_t(x,t) = \frac{x+t}{3} \Rightarrow g(x,t) = \frac{(x+t)^2}{6}$$

$$u(x,t) = -\frac{(x-2t)^2}{6} + \frac{(x+t)^2}{2} = \boxed{-\frac{t^2}{2} + tx}$$