PDE: HOMEWORK 7

Due Friday, October 28st (at the start of the recitation).

- From the Strauss textbook: 6.1.6, 6.2.1, 6.2.4, 6.2.7, 6.3.2, 6.3.3

Date: October 22, 2016.

6.1.6. Assume a radial solution, so that $u_{rr} + \frac{1}{r}u_r = 1$, and thus $(ru_r)_r = r$. Integrating and dividing by r gives $u_r = \frac{1}{2}r + c_1 r^{-1}$. Integrating again gives $u = \frac{1}{4}r^2 + c_1 \ln r + c_2$. The boundary conditions $u(a) = u(b) = 0$ imply $c_1 = \frac{b^2-a^2}{4(\ln a - \ln b)}$ and $c_2 = \frac{a^2 \ln b - b^2 \ln a}{4(\ln a - \ln b)}$.

Section 6.2

6.2.1. Guess that $u = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$. Then $u_{xx} + u_{yy} = 2A + 2C$. If u is harmonic, then $C = -A$. The boundary conditions lead to the equations

$$
\begin{aligned}
By + D &= -a \\
Bx + E &= b \\
2Aa + By + D &= 0 \\
By + 2Cb + E &= 0.
\end{aligned}
$$

The first two equations imply $B = 0$, $D = -a$ and $E = b$. Plugging these results into the last two equations gives $A = \frac{1}{2}$ and $C = -\frac{1}{2}$. Thus $u(x, y) = \frac{1}{2}(x^2 - y^2) - ax + by + F$, where F is arbitrary.

6.2.4 Let u_1 be the solution with the boundary data $(x,0,0,0)$ and u_2 be the solution with the boundary data $(0,0,0,y^2)$. Then the desired solution is given by $u = u_1 + u_2$

To ... We use separation of variables

$$u = X(x)\,Y(y) \Rightarrow -\frac{X''}{X} = \frac{Y''}{Y} = \lambda\ ,\ \lambda > 0$$

$$X'' + \lambda X = 0 \Rightarrow X = A\cos(\beta x) + B\sin(\beta x) \quad \beta^2 = \lambda$$

$$Y'' - \lambda Y = 0 \Rightarrow Y = C\cosh(\beta y) + D\sinh(\beta y)$$

$$u_x = X'(x)\,Y(y)$$

$$X' = -\beta A\sin\beta x + \beta B\cos\beta x$$

To find u_1 we apply the first boundary data

$$X'(0) = X'(1) = 0 \Rightarrow B = 0$$

$$\Rightarrow \beta = n\pi \quad n = 0, 1, 2, 3, \ldots$$

$$X(x) = A\cos(n\pi x)$$

(note that $\beta = 0$ also gives rise to an eigenfunction of X'')

$$Y(1) = C\cosh(\pi n) + D\sinh(\pi n) = 0$$

$$\lambda > 1: \; D = -C\coth(n\pi)$$

$$\lambda = 0: \; Y''(y) = 0 \Rightarrow Y = Cy + D \overset{Y(1)=0}{\Rightarrow} C = -D$$

Thus,

$$u_1(x,y) = \frac{1}{2}A_0(1-y) + \sum_{n=1}^{\infty} A_n\left(\cosh(n\lambda y) - \coth(n\pi)\sinh(n\lambda y)\right)\cos(n\pi x)$$

$$u_1(x,0) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos n\pi x = x$$

$$A_n = 2\int_0^1 x\cos n\pi x\,dx = \left.\frac{\pi n(x\sin(\pi n x)) + \cos(\pi n x)}{(\pi n)^2}\right|_{x=0}^{1}$$

$$= 2\left(\frac{\pi n\sin(\pi n)}{(\pi n)^2} + \frac{\cos(\pi n)}{(\pi n)^2} - \frac{1}{(\pi n)^2}\right) =$$

$$= \frac{2((-1)^n - 1)}{(\pi n)^2} \quad \text{when } n = 1, 2, \ldots$$

$$A_0 = 2\int_0^1 x\,dx = 1 \quad \text{when } n = 0$$

$$\boxed{u_1(x,y) = \frac{1}{2}(1-y) + \sum_{n=1}^{\infty} \frac{2((-1)^n - 1)}{(\pi n)^2}\left(\cosh(n\lambda y) - \coth(n\pi)\sinh(n\lambda y)\right)\cos(n\pi x)}$$

To find u_2, we also use separation of variables

$$u_2 = X(x)Y(y) \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = \lambda, \beta^2 = \lambda > 0$$

$$\Rightarrow Y'' + \lambda Y = 0 \Rightarrow Y = A\cos(\beta y) + B\sin(\beta y)$$

$$X'' - \lambda X = 0 \Rightarrow X = C\cosh(\beta x) + D\sinh(\beta x)$$

$$Y(0) = Y(1) = 0 \Rightarrow A = 0, \ \beta = n\pi, \ n = 1, 2, 3$$

($\lambda = 0$ is not an eigenvalue by the boundary condition)

$$X'(0) = Cn\pi\sinh(n\pi x) + Dn\pi\cosh(n\pi x) = 0$$

$$\Rightarrow D = 0$$

Therefore $X(x) = C\cosh(n\pi x)$

$$u_2(x,y) = \sum_{n=1}^{\infty} A_n \cosh(n\pi x)\sin(n\pi y)$$

$$(u_2)_x(1,y) = n\pi \sum_{n=1}^{\infty} A_n \sinh(n\pi)\sin(n\pi y) = y^2$$

$$A_n = \frac{2}{(n\pi)\sinh(n\pi)} \int_0^1 y^2 \sin(n\pi y)\,dy$$

$$= \frac{2}{(n\pi)\sinh(n\pi)} \cdot \frac{(2-\pi^2 n^2)\cos(\pi n) + 2\pi n\sin(\pi n) - 2}{\pi^3 n^3}$$

$$= \frac{2}{\sinh(n\pi)} \frac{(2-(\pi n)^2)(-1)^n - 2}{(\pi n)^4} = \frac{2}{\sinh(n\pi)}\left(-\frac{(-1)^n}{(\pi n)^2} + \frac{2((-1)^n-1)}{(\pi n)^4}\right)$$

Thus $\boxed{u_2 = \sum_{n=1}^{\infty} \frac{2}{\sinh(n\pi)}\left(-\frac{(-1)^n}{(\pi n)^2} + \frac{2((-1)^n-1)}{(\pi n)^4}\right)\cosh(n\pi x)\sin(n\pi x)}$

and $\boxed{u = u_1 + u_2}$

6.2.7.

(a) Separating variables leads to $X'' = -\lambda X$ and $Y'' = \lambda Y$ for some constant λ. The boundary conditions $X(0) = X(\pi) = 0$ imply that $X_n(x) = \sin nx$ and $\lambda_n = n^2$. The solutions of $Y'' = n^2 Y$ are $Y_n = A_n e^{-ny} + B_n e^{ny}$. The boundary condition $\lim_{y\to\infty} Y(y) = 0$ implies that $B_n = 0$. Thus the solution takes the form

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin(nx)\, e^{-ny}.$$

The boundary condition $u(x,0) = h(x)$ implies

$$h(x) = \sum_{n=1}^{\infty} A_n \sin nx,$$

so the coefficients are given by

$$A_n = \frac{2}{\pi}\int_0^{\pi} h(x)\sin nx\, dx.$$

(b) Without the condition at infinity, solutions would not be unique. They would take the form

$$u(x,y) = \sum_{n=1}^{\infty} \sin nx \left(A_n e^{-ny} + B_n e^{ny}\right),$$

where

$$A_n + B_n = \frac{2}{\pi}\int_0^{\pi} h(x)\sin nx\, dx.$$

6.3.2. In the full Fourier series for $h(\theta) = 1 + 3\sin\theta$, $A_0 = 2$, $B_1 = 3/a$ and all other coefficients are zero. Thus by equation (6.3.10), $u(r,\theta) = 1 + \frac{3}{a} r\sin\theta$. In rectangular coordinates $u(x,y) = 1 + 3y/a$.

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6.3.3 By Eq (6.3.10) in the textbook, we have

$$u = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)$$

$$u(a,\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} a^n (A_n \cos n\theta + B_n \sin n\theta)$$

$$= \sin^2\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$$

$\Rightarrow B_1 = \frac{3}{4a}$ and $B_3 = -\frac{1}{4a^3}$ (remaining coefficients are zero)

$$\Rightarrow \boxed{u = \frac{3r}{4a}\sin\theta - \frac{r^3}{4a^3}\sin 3\theta}$$