PDE: HOMEWORK 8

Due Friday, November 4th (at the start of the recitation)

- From the Strauss textbook: 6.3.1, 6.3.4
- Solve

$$\begin{cases} \Delta u = x^2 + y^2, \text{ for } \{(x,y) : x^2 + y^2 < 4\}; \\ u = 1 \text{ for } x^2 + y^2 = 4. \end{cases}$$

  Hint: write $\Delta u = u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2}$, then look for a radially symmetric solution in the form $u = ar^n$.
- Suppose $u(r,\theta)$ is a harmonic function in $|x| \leqslant 1$ satisfying $u(1,\theta) = |\cos(\theta)|$, $\theta \in [0, 2\pi)$. What is the highest number of derivatives (in r and θ) of the solution $u(r,\theta)$ at the point $(r,\theta) = (1/2, \pi/2)$.

  Hint: you do not need to solve the problem to answer this question, just use the Poisson formula representation of the solution.

Date: October 29, 2016.

Section 6.3

6.3.1.

(a) By the Maximum Principle, u attains its maximum on the boundary. Since the maximum of $3\sin 2\theta + 1$ is 4, the maximum of u on $\overline{D}$ is 4.

(b) By the Mean Value Property, the value of u at the origin is the average of $3\sin 2\theta + 1$ on the circumference. So

$$u(0,0) = \frac{1}{2\pi}\int_0^{2\pi} 3\sin 2\theta + 1\, d\theta = 1.$$

6.3.4. By a straightforward computation, we get

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{a^2-r^2}{a^2-2ar\cos(\theta-\varphi)+r^2}\right)=0$$

Additional Problems

1. Solve $\begin{cases} \Delta u = x^2+y^2 & \text{for } \{(x,y) : x^2+y^2<4\} \\ u=1 & \text{for } x^2+y^2=4 \end{cases}$

Solution: If we assume $u = ar^n$, we have

$$u_r = anr^{n-1} \qquad u_{\theta\theta} = 0$$

$$u_{rr} = an(n-1)r^{n-2}$$

Therefore

$$\Delta u = u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} = r^2$$

$$\Rightarrow an(n-1)r^{n-2} + anr^{n-2} = r^2$$

$$an^2 r^{n-2} = r^2 \Rightarrow n = 4 \checkmark$$

From $x^2+y^2=r^2=4$ we get

$$u(2) = a\cdot 2^4 = 1 \Rightarrow a = \frac{1}{16}$$

Therefore $u(r,\theta) = \frac{1}{16}r^4$

2. $u(r,\theta)$ is a harmonic function in $|x| \leq 1$ satisfying $u(1,\theta) = |\cos\theta|$ for $\theta \in [0, 2\pi)$. What is the highest number of derivatives in r and θ of the solution $u(r,\theta)$ at the point $(r,\theta) = \left(\frac{1}{2}, \frac{\pi}{2}\right)$

Soln By Strauss' Eq (6.3.13)

$$u(r,\theta) = \left(1 - \frac{r^2}{1}\right) \int_0^{2\pi} \frac{|\cos\varphi|}{1 - 2r\cos\left(\frac{\pi}{2} - \theta\right) + \frac{1}{4}} \frac{d\varphi}{2\pi}$$

since the denominator of the integrand is non-zero for any θ, r in a neighbourhood of $\left(\frac{1}{2}, \frac{\pi}{2}\right)$, the integrand is a C^∞ function w/r/t θ and r at that point

Thus $u(r,\theta)$ is also C^∞ there