PDE: HOMEWORK 9

Due Friday, November 11th (at the start of the recitation)

- From the Strauss textbook: $\boxed{3.1.1, 7}$.1.2, 7.1.5, 7.1.7
- Solve

$$\begin{cases} u_t - u_{xx} = x^2, \text{ for } -\infty < x < \infty;\ 0 < t < \infty \\ u(x,0) = e^x - \frac{x^4}{12} \end{cases}$$

Hint: Look for a solution in the form $v(x,t) - \frac{x^4}{12}$ where v is the solution to the corresponding homogeneous equation with the initial condition $v(x,0) = e^x$.

Date: November 5, 2016.

3.1.1 by (3.1.6) in Strauss.

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_0^\infty \left[e^{-\frac{(x-y)^2}{4kt}} - e^{-\frac{(x+y)^2}{4kt}}\right]e^{-y}\,dy$$

$$\frac{-(x-y)^2}{4kt} - y = -\frac{1}{4kt}\left[y^2 - 2xy + 4kty + x^2\right]$$

$$= -\frac{1}{4kt}\left[(y+2kt-x)^2 + x^2 - (2kt-x)^2\right]$$

$$= -\frac{1}{4kt}\left[(y+2kt-x)^2 + 4ktx - 4k^2t^2\right]$$

$$= -\frac{1}{4kt}(y+2kt-x)^2 + kt - x.$$

$$-\frac{(x+y)^2}{4kt} - y = -\frac{1}{4kt}(y+2kt+x)^2 + kt + x$$

$$\Rightarrow u(x,t) = \frac{e^{kt-x}}{\sqrt{4\pi kt}}\int_0^\infty e^{-\frac{(y+2kt-x)^2}{4kt}}\,dy - \frac{e^{kt+x}}{\sqrt{4\pi kt}}\int_0^\infty e^{-\frac{(y+2kt+x)^2}{4kt}}\,dy$$

Let $\begin{cases} p = \dfrac{y+2kt-x}{\sqrt{4kt}}, & dy = \sqrt{4kt}\,dp. \\ q = \dfrac{y+2kt+x}{\sqrt{4kt}}, & dy = \sqrt{4kt}\,dq. \end{cases}$

$$\Rightarrow u(x,t) = \frac{e^{kt-x}}{\sqrt{\pi}}\int_{\frac{2kt-x}{\sqrt{4kt}}}^\infty e^{-p^2}\,dp - \frac{e^{kt+x}}{\sqrt{\pi}}\int_{\frac{2kt+x}{\sqrt{4kt}}}^\infty e^{-q^2}\,dq$$

$$= \frac{e^{kt-x}}{\sqrt{\pi}}\left(\overbrace{\int_0^\infty e^{-p^2}\,dp}^{\frac{\sqrt{\pi}}{2}} - \int_0^{\frac{2kt-x}{\sqrt{4kt}}} e^{-p^2}\,dp\right) - \frac{e^{kt+x}}{\sqrt{\pi}}\left(\int_0^\infty e^{-q^2}\,dq - \int_0^{\frac{2kt+x}{\sqrt{4kt}}} e^{-q^2}\,dq\right)$$

$$= \frac{e^{kt-x}}{2} - \frac{e^{kt-x}}{\sqrt{\pi}}\int_0^{\frac{2kt-x}{\sqrt{4kt}}} e^{-p^2}\,dp - \frac{e^{kt+x}}{2} + \frac{e^{kt+x}}{\sqrt{\pi}}\int_0^{\frac{2kt+x}{\sqrt{4kt}}} e^{-q^2}\,dq$$

Chapter 7

Section 7.1

7.1.2. Suppose D is a bounded domain and u and v satisfy $\Delta u = \Delta v = f$ on D and $\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = g$ on bdy D. Then $w = u - v$ is harmonic on D and $\frac{\partial w}{\partial n} = 0$ on bdy D. By Green's First Identity

$$\iint_{\text{bdy } D} w \frac{\partial w}{\partial n}\, dS = \iiint_D \nabla w \cdot \nabla w \, d\mathbf{x} + \iiint_D w \Delta w \, d\mathbf{x}.$$

By the boundary condition and the fact that w is harmonic, this reduces to

$$\iiint_D |\nabla w|^2 \, d\mathbf{x} = 0,$$

so by the Vanishing Theorem, $|\nabla w(x,y,z)|^2 = 0$ for all $(x,y,z) \in D$. Thus w is constant, so solutions are unique up to constants.

7.1.5. Suppose $E[u] \le E[w]$ for all functions w on D. Let v be any function on D. Then

$$\mathscr{E}(\epsilon) = E[u + \epsilon v] = \frac{1}{2} \iiint_D |\nabla u + \epsilon \nabla v|^2 \, d\mathbf{x} - \iint_{\text{bdy } D} h(u + \epsilon v)\, dS$$

has a local minimum at $\epsilon = 0$, and therefore

$$0 = \mathscr{E}'(0) = \iiint_D \nabla u \cdot \nabla v \, d\mathbf{x} - \iint_{\text{bdy } D} hv \, dS = \iint_{\text{bdy } D} \left(\frac{\partial u}{\partial n} - h \right) v \, dS - \iiint_D v \Delta u \, d\mathbf{x}.$$

Now let D' be any strict subdomain of D and let $v = 1$ on D' on $v = 0$ on $D - D'$. Then

$$\iiint_{D'} \Delta u \, d\mathbf{x} = 0,$$

and since this holds for all D', the Second Vanishing Theorem implies $\Delta u = 0$ on D. This then implies

$$\iint_{\text{bdy } D} \left(\frac{\partial u}{\partial n} - h \right) v \, dS = 0$$

for all functions v. Choosing v to be a function that is equal to $\frac{\partial u}{\partial n} - h$ on bdy D gives

$$\iint_{\text{bdy } D} \left(\frac{\partial u}{\partial n} - h \right)^2 dS = 0,$$

so $\frac{\partial u}{\partial n} = h$ on bdy D.

7.1.7. Let $w = w_0 + c_1 w_1 + \cdots + c_n w_n$, and define

$$F(c_1, \ldots, c_n) = E(w) = \frac{1}{2} \iiint_D |\nabla w_0 + c_1 \nabla w_1 + \cdots + c_n \nabla w_n|^2 \, d\mathbf{x}.$$

For the choice of coefficients which minimizes F, the partial of F with respect to each c_j must vanish. Hence

$$0 = \frac{\partial F}{\partial c_j} = \iiint_D (\nabla w_0 + c_1 \nabla w_1 + \cdots + c_n \nabla w_n) \cdot \nabla w_j \, d\mathbf{x},$$

or in terms of the inner product,

$$(\nabla w_0, \nabla w_j) + \sum_{k=1}^{n} c_k (\nabla w_k, \nabla w_j) = 0.$$

By symmetry of the inner product this proves the desired result.

Solve $\begin{cases} u_t - u_{xx} = x^2 & \text{for } -\infty < x < \infty,\ 0 < t < \infty \\ u(x,0) = e^x - \frac{x^4}{12} \end{cases}$

<u>Solution</u> Let $u(x,t) = v(x,t) - \frac{x^4}{12}$.

Since $u_t - u_{xx} = v_t - v_{xx} + x^2 = x^2$

v is the solution to

$$\begin{cases} v_t - v_{xx} = 0 \\ v(x,0) = e^x \end{cases}$$

Here $v = e^{x+t}$ and therefore $u = e^{x+t} - \frac{x^4}{12}$