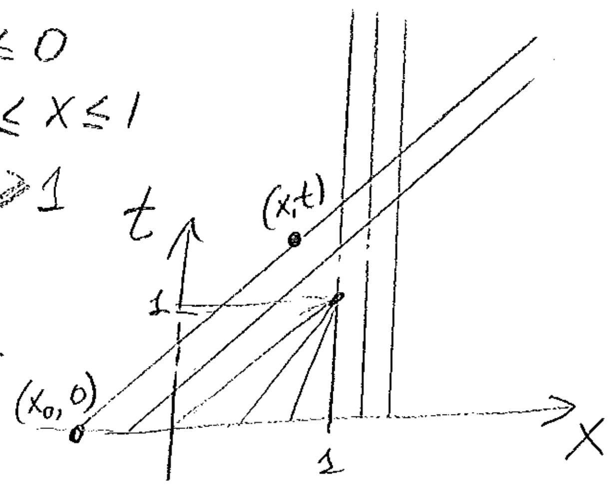
Evans

§3.4.1, Example 1 (Shock waves) Let us consider the initial-value problem for Burger's equation

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ u = g & \text{on } \mathbb{R} \times \{t=0\} \end{cases}$$

with the initial data

$$g(x) = \begin{cases} 1 & \text{if } x \le 0 \\ 1-x & \text{if } 0 \le x \le 1 \\ 0 & \text{if } x \geq 1 \end{cases}$$



Solution: Following Ex 3 in Strauss §14.1, we determine the characteristic lines and their intersection:

$$0 = u_t + u u_x = (u_t, u_x) \cdot \begin{pmatrix} 1 \\ u \end{pmatrix}$$ i.e directional derivative in the direction u is zero.

Therefore, the characteristic curves $(x(t), t)$ have slope u, which is given by the solution of the ODE:

$$(*) \quad \frac{dx}{dt} = u(x,t).$$ Note that u is constant along those curves:

$$\frac{d}{dt}\left[u(x(t),t)\right] = u_t + \underbrace{\frac{dx}{dt}}_{u \text{ (*)}} u_x = u_t + u u_x = 0$$

Therefore the characteristic curves are indeed lines.

$$\frac{x-x_0}{t-0} = \frac{dx}{dt} = u(x,t) = u(x,0) = g(x_0)$$

For $x_0<0$: $g(x_0)=1$, and therefore the characteristic curves are given by $\frac{x-x_0}{t} = 1 \Rightarrow x-x_0 = 1\cdot t$

$$x_0 = x-1\cdot t$$

$$u(x,t) = g(x_0) = 1$$

For $0 \le x_0 \le 1$: $\frac{x-x_0}{t} = 1-x_0$

$$t(1-x_0) = x-x_0$$

$$t-tx_0 = x-x_0$$

$$t-x = x_0(t-1)$$

$$x_0 = \frac{t-x}{t-1}$$

$$u(x,t) = g(x_0) = 1-x_0 = 1-\frac{t-x}{t-1} = \frac{1-x}{1-t}$$

For $x_0>1$: $g(x_0)=0$, and therefore the characteristic lines are given by $\frac{x-x_0}{t} = 0 \Rightarrow x-x_0=0$

$$u(x,t) = g(x_0) = 0$$

(a) Classical solution:

For $t\le 1$, the solution is given by

$$u(x,t) = \begin{cases} 1, & x<t \\ \frac{1-x}{1-t}, & t<x<1 \\ 0, & x>1 \end{cases}$$

(6) Weak solution:

These curves intersect at $t=1$, and for $t \geq 1$, the different characteristic lines define u differently. Therefore u is not well-defined in the classical sense. Accordingly, we look for a weak solution for $t \geq 1$.

Let $A'(u) = u \Rightarrow A = \frac{u^2}{2}$

$$u_t + A(u)_x = u_t + \frac{dA}{du} \cdot u_x = u_t + u u_x = 0$$

$$\int_0^\infty \int_{-\infty}^\infty [u \varphi_t + \frac{u^2}{2} \varphi_x] \, dx \, dt = 0$$

for all test functions $\varphi(x,t)$ defined in the half-plane.

The initial condition for $x < 1$ suggests that $u^+ = 0$ while the initial condition for $x < 0$ suggests that $u^- = 1$.

By Rankine-Hugoniot, the slope of the $x = \xi(t)$ is given by

$$\frac{A(u^+) - A(u^-)}{u^+ - u^-} = \frac{\frac{0^2}{2} - \frac{1^2}{2}}{0 - 1} = \frac{1}{2}$$

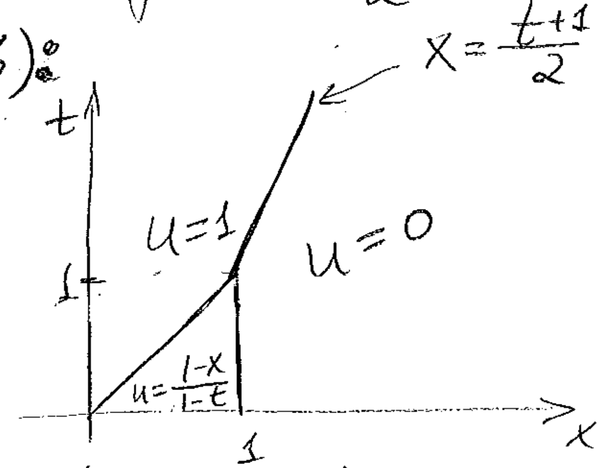
since $x = \xi(t)$ must also contain $(1,1)$

we have $x = \xi(t) = \frac{t+1}{2}$

Therefore, for $t \geq 1$, the weak solution is given by

$$u(x,t) = \begin{cases} 1 & \text{if } x < \frac{t+1}{2} \\ 0 & \text{if } x > \frac{t+1}{2} \end{cases}$$

Combining (a) and (b):



Here is the dynamics of the shock