PDE Recitation 4 notes

1) solve the wave eq. w/ homogeneous Dirichlet conditions

$$u_{tt} = u_{xx} \quad \text{for } 0 < x < \pi$$
$$u(0,t) = 0 = u(\pi, t)$$
$$u(x,0) = \phi(x) = \sin^3 x$$
$$u_t(x,0) = \psi(x) = -\sin x + 5 \sin 4x + \sin 7x$$

Soln: By trig identities

$$\sin^3 x = \frac{3 \sin x - \sin 3x}{4}$$

$$u(x,t) = \left(\frac{3}{4} \cos t - \sin t\right) \sin x - \frac{1}{4} \cos 3t \sin 3x + \frac{5}{4} \sin 4t \sin 4x + \frac{1}{7} \sin 7t \sin 7x$$

2) Wave equation with a restoration term (e.g. restoring force on the medium proportional to the displacement of the particle from its equilibrium position — 1D string, 2D - drum head)

$$\begin{cases} u_{tt} = u_{xx} - u & 0 < x < l,\ t \geq 0 \\ u(0,t) = u(l,t) = 0 & t \geq 0 \\ u(x,0) = f(x) & 0 \leq x \leq l \\ u_t(x,0) = 0 \end{cases}$$

Soln: We look for solutions of the form

$u(x,t) = X(x)T(t)$

$XT'' = X''T - XT$ divide by XT

$$\frac{T''}{T} = \frac{X''}{X} - 1$$

$$\frac{T''}{T} + 1 = \frac{X''}{X} = -\lambda$$ (by the discussion on p. 85 of the text $\lambda > 0$

$$\sum = \lambda_n = \left(\frac{n\pi}{l}\right)^2 \quad X_n(x) = \sin\frac{n\pi x}{l}$$

We solve the ODE given by

$$T'' + (\lambda_n + 1)T = 0$$

By ODE theory $x^2 + (\lambda + 1) = 0 \Rightarrow x = \sqrt{\lambda_n + 1} = \beta_n$

$$T_n(t) = A_n \cos(\beta_n t) + B_n \sin(\beta_n t)$$

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos(\beta_n t) + B_n \cos(\beta_n t)\right) \sin\frac{n\pi x}{l}$$

$$0 = u_t(x,0) = \sum_{n=1}^{\infty} \left(-A_n \sin \beta_n t - B_n \cos(\beta_n t)\right) \sin \frac{n\pi x}{l} =$$

$$\Longrightarrow B_n = 0$$

Therefore $u(x,t) = \sum_{n=1}^{\infty} A_n \cos(\beta_n t) \sin \frac{n\pi x}{l}$

given IC

$$f(x) = \sum_n A_n \sin\left(\frac{n\pi x}{l}\right)$$

3) $\int$ Gauss 4.2.1