PDE: PRACTICE PROBLEMS 1

Instructions: Please try to solve these problems without textbooks, notes, devices or materials of any kind. Please provide require a written explanation as well as an answer all questions. Your grade on the actual exam largely depends on showing *correct* and *complete* reasoning. You may assume basic facts about arithmetic or calculus and basics results from class, but please always explain which facts, results or ideas you are using, either by giving their name or by including a precise statement of what they say.

Question 1. Find the function $u(x, y)$ satisfying

$$\begin{aligned} 2u_x + u_y &= x, \\ u(x, 0) &= x^2 \end{aligned}$$

Question 2. Consider the PDE for $u = u(x, y)$

$$y^2 u_x + x^2 u_y = 0, \ (x, y) \in \mathbb{R}^2$$

(a) Find an equation for the characteristic curves of the PDE.
(b) Use your answer to part (a) to derive the general form of the solution.

Question 3. Let $u = u(x, t)$ solve the wave equation on the real line,

$$\begin{cases} u_{tt} - v_{xx} = 0, t > 0, x \in \mathbb{R} \\ u(x, 0) = u_0(x) \\ u_t(x, 0) = 0, \end{cases}$$

with initial data

$$u_0(x) = \begin{cases} (x-2)^5 & x \in [2, \infty) \\ 0 & x \in (-\infty, 2) \end{cases}$$

What is the highest order of derivatives of u? Why?

Question 4. Consider the initial value problem

$$\begin{aligned} u_t + u &= u_{xx}, \ x \in \mathbb{R}, t \geqslant 0 \\ u(x, 0) &= e^{-x}, \ x \in \mathbb{R} \end{aligned}$$

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(a) Derive an equation satisfied by $v(x,t) = e^t u(x,t)$.
(b) Solve the original initial value problem.

Question 5. Find a bounded solution $u = u(t,x)$ of the heat equation

$$\begin{cases} u_t = u_{xx}, \ \ 0 \leqslant x \leqslant \pi \\ u(x,0) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^2} \\ u(0,t) = u(\pi,t) = 0 \end{cases}$$

Question 6. From the Strauss textbook: 5.1.2, 5.1.4.

Question 7. Consider the heat equation $u_t = u_{xx}$ on the bounded domain $0 \leqslant x \leqslant 1$ with boundary data $u(0,t) = 0$ and $u(1,t) = 1$. Find a solution satisfying the initial condition $u(x,0) = x^2$.