Name | Model Solutions

PDE, Fall 2016
Quiz #2

You are not allowed to use books or notes. The quiz is graded out of 40 points (+5 bonus points). Please show all your work. No credit is given for unsupported answers.

1. (15 points) Let $u(x,y)$ solve $\Delta u = 0$ in $(x,y) \in (0,1) \times (0,1)$ with

$$\begin{cases} u(0,y) = y(1-y); \\ u(1,y) = -1; \\ u(x,0) = -x; \\ u(x,1) = -x^2. \end{cases}$$

Find the maximum and minimum values of u in $(x,y) \in [0,1] \times [0,1]$. Please justify your answer.

since u is harmonic, it's max and min values are achieved on the boundary. Thus, $u(0,y) = y(1-y)$ is maximized @ $y = \frac{1}{2}$ (other boundary values nonpositive)

Therefore $\max\limits_{(x,y)\in(0,1)\times(0,1)} u = u(0,\frac{1}{2}) = \frac{1}{4}$ and

$\min\limits_{x\in(0,1)\times(0,1)} u = u(1,y) = -1.$

2. (30 points) Solve by separation of variables

$$\begin{cases} \Delta u = 0, \ (x,y) \in [0,1] \times [0,\pi]; \\ u(x,0) = u(x,\pi) = 0; \\ u(0,y) = 0, u(1,y) = 2\cos 2y - 2. \end{cases}$$

Hint: when calculating the coefficients in sine series on $[0,l]$, you may use the formula $A_n = \frac{2}{l}\int_0^l \phi(x)\sin\frac{n\pi x}{l}dx$ and the identity $2\cos a \sin b = \sin(a+b) - \sin(a-b)$.

we look for a separated solution $u(x,y) = X(x)Y(y)$

$$\frac{Y''}{Y} = -\frac{X''}{X} = -\lambda, \text{ where } \lambda > 0 \text{ by the boundary condition}$$

$$\begin{cases} Y'' + \lambda Y = 0 \xRightarrow{Y(0)=0} y = \sin(ny), \ n = 1, 2 \ldots; \ n^2 = \lambda_n \\ X'' - \lambda X = 0 \Rightarrow X = A_n \sinh(nx) \end{cases}$$

$X(0) = 0$ (cosh terms vanish)

$$u(1,y) = \sum_{n=1}^{\infty} A_n \sinh(n)\sin(ny) = 2\cos 2y - 2$$

$$A_n \sinh(n) = \frac{2}{\pi}\int_0^{\pi}(2\cos 2y - 2)\sin(ny)\,dy = \frac{2}{\pi}\int_0^{\pi} \sin((2+n)y) - \sin((2-n)y) - 2\sin(ny)\,dy$$

$n \neq 2$
$$= \frac{2}{\pi}\left[-\frac{\cos((2+n)y)}{2+n} + \frac{\cos((2-n)y)}{2-n} + \frac{2\cos(ny)}{n}\right]_{y=0}^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{(-1)^n - 1}{2+n} + \frac{(-1)^n - 1}{2-n} + \frac{2(-1)^n - 2}{n}\right]$$

n odd
$$= \frac{2}{\pi}\left[\frac{2}{2+n} - \frac{2}{2-n} - \frac{4}{n}\right] = \frac{-32}{\pi(4-n^2)n} \quad (A_n = 0 \text{ for } n \text{ even})$$

$$\left[(n=2) = \frac{2}{\pi}\left[-\frac{(-1)^2 - 1}{2+2} + \frac{2(-1)^2 - 2}{2}\right] = 0\right]$$

$$\boxed{u(x,y) = \sum_{\substack{n \text{ odd} \\ \text{positive}}} \frac{-32}{\pi(4-n^2)n} \cdot \frac{\sinh(nx)}{\sinh(n)} \sin(ny)}$$