WEEK 10

Problem 1 (Putnam 2001 A1)**.** Consider a set S and a binary operation $*$, i.e., for each $a, b \in S$, $a * b \in S$. Assume $(a * b) * a = b$ for all $a, b \in S$. Prove that $a * (b * a) = b$ for all $a, b \in S$.

Problem 2 (Missouri Collegiate Competition 2003)**.** Let $x_1 > 1$ be odd and define the sequence $\{x_n\}_{n=1}^{\infty}$ recursively by $x_n = x_{n-1}^2 - 2$, $n \geqslant 2$. Prove that for any pair of integers j, k satisfying $1 \leqslant j < k$, the terms x_j, x_k are relatively prime.

Problem 3 (Putnam 1989 B2)**.** Let S be a non-empty set with an associative operation that is left and right cancellative ($xy = xz$ implies $y = z$, and $yx = zx$ implies $y = z$). Assume that for every a in S the set $\{a^n : n = 1, 2, 3, \ldots\}$ is finite. Must S be a group?

Problem 4 (Leo Schneider 2017)**.** Two points are chosen uniformly at random from the unit interval $[0, 1]$. What is the probability that the points will be within a distance of $1/8$ of each other?

Problem 5 (Missouri Collegiate Competition 2006)**.** The array below is called a magic square because the sum of the three numbers along any row, any column, or the two diagonals, is the same (namely, 15).

8	1	6
3	5	7
4	9	2

(a) Construct a 3×3 multi-magic square, that is, a 3×3 array of 9 distinct integers such that the PRODUCT of the three numbers along any row, any column, or the two diagonals, is the same.
(b) Show that no multi-magic square can be constructed with nine *consecutive* integers.

Problem 6 (Putnam 2004 - A6)**.** Suppose that $f(x, y)$ is a continuous real-valued function on the unit square $0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1$. Show that

$$\int_0^1 \left(\int_0^1 f(x,y)dx \right)^2 dy + \int_0^1 \left(\int_0^1 f(x,y)dy \right)^2 dx$$
$$\leqslant \left(\int_0^1 \int_0^1 f(x,y)dx\,dy \right)^2 + \int_0^1 \int_0^1 [f(x,y)]^2\,dx\,dy.$$

Date: November 6, 2025.