WEEK 11

Problem 1 (Putnam 1972 A3)**.** Let S be a set and $*$ be a binary operation on S satisfying the laws (i) $x * (x * y) = y$ and (ii) $(y * x) * x = y$ for all $x, y \in S$. Prove that $*$ is commutative but not necessarily associative.

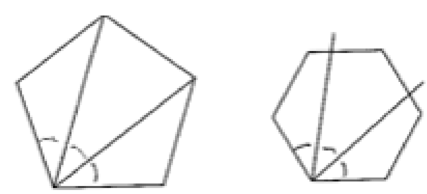
Problem 2 (Putnam 1972 B3)**.** Let A and B be two elements in a group such that $ABA = BA^2B$, $A^3 = 1$ and $B^{2n-1} = 1$ for some positive integer n. Prove that $B = 1$.

Problem 3 (Putnam 1977 B6)**.** Let H be a subgroup with h elements in a group G. Suppose that G has an element a such that for all x in H, $(xa)^3 = 1$, the identity. In G, let P be the subset of all products

$$x_1ax_2a \dots x_na$$

with n a positive integer and the x_i's in H. (a) Show that P is a finite set. (b) Show that in fact P has no more than $3h^2$ elements.

Problem 4 (Missouri Collegiate Competition 2005)**.** Suppose that f is a polynomial of positive degree n with integer coefficients. Prove that there are infinitely many integers x for which $f(x)$ is composite. (Here, composite means those integers, positive or negative, whose absolute value is not 1 or a prime; thus, -4 is composite while 1 and -2 are not.)



Problem 5 (Lehigh Math Competition 2006)**.** The angle trisectors of a regular pentagon intersect other vertices of the pentagon; i.e., they are diagonals of the pentagon (see the left side of the figure). What is the smallest $n > 5$ such that the angle trisectors of a regular n-gon intersect other vertices of the n-gon? (The right side of the figure illustrates that $n = 6$ does not work.)

Problem 6 (Missouri Collegiate Competition 2005)**.** Suppose that $f\colon [0, \infty) \to [0, \infty)$ is a differentiable function with the property that the area under the curve $y = f(x)$ from $x = a$ to $x = b$ is equal to the arclength of the curve $y = f(x)$ from $x = a$ to $x = b$. Given that $f(0) = 5/4$, and that $f(x)$ has a minimum value on the interval $(0, \infty)$, find that minimum value.

Date: November 13, 2025.