WEEK 3

Problem 1 (Missouri Collegiate Competition)**.** Suppose f is a continuous real-valued function on the interval $[0, 1]$. Show that

$$\int_0^1 x^2 f(x)\, dx = \frac{1}{3} f(\xi)$$

for some $\xi \in [0, 1]$.

Problem 2 (Lehigh)**.** Let $A = \sin x + \cos x$. Write $\sin^4 x + \cos^4 x$ as a polynomial in A.

Problem 3 (Leo Schneider Student Team Competition)**.** Determine the number of three word phrases that can be formed from the letters in MATH ALL DAY. No "words" can be empty, and words do not have to make sense. For example, MH DATA LLAY and T DMALL YAAH are valid phrases, but not THAL LAMADY. You do not have to simplify your answer. (Bonus: What's the best phrase you can come up with?)

Problem 4 (Missouri Collegiate Competition)**.** For which positive integers a does there exist a right triangle with integer sides, at least one of which is a?

Problem 5 (Lehigh)**.** What is the largest positive integer n such that n is divisible by every positive integer m which satisfies $m^2 + 4 \leqslant n$?

Problem 6 (Lehigh)**.** Eight points on the circumference of a circle are chosen and all $\binom{8}{2} = 28$ chords connecting them are drawn. It turns out that no three chords intersect in the same point in the interior of the circle. Into how many regions do these chords divide the interior of the circle?

Problem 7 (Missouri Collegiate Competition)**.** Find all real solutions to the equation $4x^2 - 40\lfloor x \rfloor + 51 = 0$. (Note that $\lfloor x \rfloor$ denotes the floor function, the greatest integer less than or equal to x.)

Problem 8 (Putnam)**.** Define a selfish set to be a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $\{1, 2, \ldots, n\}$ which are minimal selfish sets, that is, selfish sets none of whose proper subsets is selfish.

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