WEEK 4

Problem 1 (Lehigh)**.** Let $f(x) = \frac{x}{x+3}$ for $x \neq -3$. List all values of x for which $f(f(x)) = x$.

Problem 2 (Lehigh 2014)**.** How many ordered 4-tuples of nonnegative integers satisfy $a + b + c + d \leqslant 15$?

Problem 3 (Missouri Collegiate Competition 2015)**.** Consider the following two-person game: Start with 2015 pennies on a table. Players A and B alternate turns, with A going first. A legal move consists of removing any divisor of the number of pennies on the table, as long as the divisor is *strictly less than* the number of pennies on the table. For example, at the start of the game, player A can remove 1, 5, 13, 31, 65, 155, or 403 pennies, but not 2015 pennies. If player A removes 5 pennies, then player B could remove, say, 3 pennies, etc.
The game ends when no legal move is possible (i.e., when only one penny remains), and whoever's turn it is loses (and the opponent wins). So the objective is to leave your opponent with just one penny. Is there a winning strategy? If so, who wins, A or B?

Problem 4 (Missouri Collegiate Competition 2015)**.** Show that for each positive integer n, the function $f_n(x) = x^n + (x-1)^n - (x+1)^n$ has a unique nonzero real root r_n and that $r_n \leqslant r_{n+1}$ for all n.

Problem 5 (2002 USAMO)**.** Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 - y^2) = xf(x) - yf(y).$$

Problem 6 (Putnam 2016-B1)**.** Let $x_0, x_1, x_2, \ldots$ be the sequence such that $x_0 = 1$ and for $n \geqslant 0$,

$$x_{n+1} = \ln(e^{x_n} - x_n)$$

(as usual, the function ln is the natural logarithm). Show that the infinite series

$$x_0 + x_1 + x_2 + \cdots$$

converges and find its sum.

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