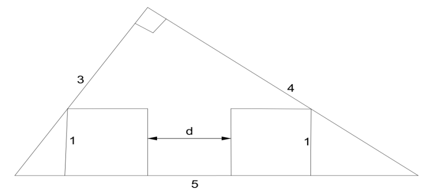
WEEK 5

Problem 1 (2013 Leo Schneider Student Team Competition)**.** Two unit squares stand on the hypotenuse of a $(3, 4, 5)$ triangle in such away that they line inside the triangle, and a corner of one touches the side of length 3 and a corner of the other touches the side of length 4, as shown in the figure to the right. What is the distance d between the squares?



Problem 2 (1972 USAMO)**.** A random number generator randomly generates integers from the set $\{1, 2, ..., 9\}$ with equal probability. Find the probability (with explanation) that after n numbers are generated, their product is a multiple of 10.

Problem 3 (2013 Leo Schneider Student Team Competition)**.** Let A and B be 3×3 matrices with integer entries, such that $AB = A + B$. Find all possible values of $\det(A - I)$. Note: The symbol I represents the 3×3 identity matrix.

Problem 4 (Putnam 1995 A-1)**.** Let S be a set of real numbers which is closed under multiplication (that is, if a and b are in S, then so is ab). Let T and U be disjoint subsets of S whose union is S. Given that the product of any *three* (not necessarily distinct) elements of T is in T and that the product of any three elements of U is in U, show that at least one of the two subsets T, U is closed under multiplication.

Problem 5 (Putnam 1995-B1)**.** For a partition π of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, let $\pi(x)$ be the number of elements in the part containing x. Prove that for any two partitions π and π', there are two distinct numbers x and y in $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ such that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$. [A *partition* of a set S is a collection of disjoint subsets (parts) whose union is S.]

Problem 6 (Missouri Collegiate Competition 1998)**.** Sum the series

$$\sum_{i=1}^{\infty} \frac{36i^2 + 1}{(36i^2 - 1)^2}$$

Problem 7 (Putnam 1997 A-3)**.** Evaluate

$$\int_0^{\infty} \left(x - \frac{x^3}{2} + \frac{x^5}{2 \cdot 4} - \frac{x^7}{2 \cdot 4 \cdot 6} + \cdots \right) \left(1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots \right) dx.$$

Problem 8 (Putnam 1997 B-1)**.** Let $\{x\}$ denote the distance between the real number x and the nearest integer. For each positive integer n, evaluate

$$F_n = \sum_{m=1}^{6n-1} \min(\{\frac{m}{6n}\}, \{\frac{m}{3n}\}).$$

(Here $\min(a, b)$ denotes the minimum of a and b.)

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