WEEK 6

Problem 1 (Lehigh 2000). Evaluate

$$3+\frac{1}{4+\frac{1}{3+\frac{1}{4+\dots}}}$$

Problem 2 (Missouri Collegiate Competition 2001). Let $\{x_i\}$ denote any finite sequence with the following properties:

(a) $x_i \in \{-2, 1, 2\}$ for each x_i,
(b) $\sum_i x_i = 29$,
(c) $\sum_i x_i^2 = 59$. In considering the family of all such sequences, let $M = \max\{\sum_i x_i^3\}$ and $m = \min\{\sum_i x_i^3\}$. Determine M/m.

Problem 3 (Missouri Collegiate Competition 1998). Let I be the $n\times n$ identity matrix. Prove that $AB-BA \neq I$ for any $n \times n$ matrices A and B.

Problem 4 (Lehigh 1997). Let

$$x_1 < x_2 < x_3 < \cdots$$

be a listing of all numbers which can be written as a sum of one or more distinct factorials of positive integers $(1! = 1, 2! = 2, 3! = 6, 4! = 24...)$. So for example, $x_1 = 1, x_2 = 2, x_3 = 2 + 1, x_4 = 6$. What is x_{100}?

Problem 5 (USAMO 2005). Determine all composite positive integers n for which it is possible to arrange all divisors of n that are greater than 1 in a circle so that no two adjacent divisors are relatively prime.

Problem 6 (Missouri Collegiate Competition 2013). Neither Mathematica nor Maple can find the exact value of the following definite integral. Can you? We think you can. Do it!

$$\int_0^2 (3x^2 - 3x + 1)\cos(x^3 - 3x^2 + 4x - 2)\,dx$$

Problem 7 (Putnam 2000 B2). Prove that the expression

$$\frac{gcd(m,n)}{n}\binom{n}{m}$$

is an integer for all pairs of integers $n \geqslant m \geqslant 1$.

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