

WEEK 7

Problem 1 (Classic quant interview question). Suppose a mother has 2 children. Assume having a girl or boy is equally likely, and being born on each day of the week is equally likely.

- (1) If the mother says at least one of my children is a boy what is the probability that both children are boys?
- (2) If the mother says at least one of my children is a boy born on a Monday, what is the probability both children are boys?
- (3) If she says her youngest child is a boy, what is the probability both are boys?

Problem 2 (Lehigh '02). What is the largest number of distinct positive integers such that no four of them have sum divisible by 4?

Problem 3 (Lehigh '07). Find k so that the solutions of $x^3 - 3x^2 + kx + 8 = 0$ are in arithmetic progression.

Problem 4 (Lehigh '02). What is the number of solutions (x, y) , with x and y nonnegative integers, of the equation $4^x - 9^y = 55$?

Problem 5 (Putnam 2000-A1). Let A be a positive real number. What are the possible values of $\sum_{j=0}^{\infty} x_j^2$, given that $x_0, x_1, \dots$ are positive numbers for which $\sum_{j=0}^{\infty} x_j = A$?

Problem 6 (Putnam 1999-A2). Let $p(x)$ be a polynomial that is nonnegative for all real x . Prove that for some k , there are polynomials $f_1(x), \dots, f_k(x)$ such that

$$p(x) = \sum_{j=1}^k (f_j(x))^2.$$

Problem 7 (Missouri Collegiate Competition 2010). Consider the Diophantine equation

$$x(2x^2 + 3x + 3) = y^3 - 1.$$

Prove that it does not have a solution (x, y) in positive integers, or find such a solution if it does.

Problem 8 (USAMO 2006). Find all positive integers n such that there are $k \geq 2$ positive rational numbers $a_1, a_2, \dots, a_k$ satisfying $a_1 + a_2 + \dots + a_k = a_1 \cdot a_2 \cdot \dots \cdot a_k = n$.

Problem 9 (Putnam 2004-A1). Basketball star Shanille O'Keal's team statistician keeps track of the number, $S(N)$, of successful free throws she has made in her first N attempts of the season. Early in the season, $S(N)$ was less than 80% of N , but by the end of the season, $S(N)$ was more than 80% of N . Was there necessarily a moment in between when $S(N)$ was exactly 80% of N ?