WEEK 8

Problem 1 (Matthew Stone)**.** The $\{x_n\}$ be the following sequence involving sums of square roots of squares:

$$x_1 = \sqrt{1}, \quad x_2 = \sqrt{1+\sqrt{4}}, \quad x_3 = \sqrt{1+\sqrt{4+\sqrt{9}}}, \quad x_4 = \sqrt{1+\sqrt{4+\sqrt{9+\sqrt{16}}}}$$

and so on. Prove that $\lim_{n\to\infty x_n}$ exists. You do NOT need to determine its value.

Problem 2 (Matthew Stone)**.** Want a harder one? The $\{x_n\}$ be the following sequence involving sums of square roots of powers:

$$x_1 = \sqrt{1}, \quad x_2 = \sqrt{1+\sqrt{2^2}}, \quad x_3 = \sqrt{1+\sqrt{2^2+\sqrt{3^3}}}, \quad x_4 = \sqrt{1+\sqrt{2^2+\sqrt{3^3+\sqrt{4^4}}}}$$

and so on. Prove that $\lim_{n\to\infty x_n}$ exists. You do NOT need to determine its value.

Problem 3 (Missouri Collegiate Competition 2003)**.** For a real 2×2 matrix

$$\mathbf{X} = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$$

let $\|\mathbf{X}\| = x^2 + y^2 + z^2 + t^2$, and define a distance function by $d(\mathbf{X}, \mathbf{Y}) = \|\mathbf{X} - \mathbf{Y}\|$. Let $\Sigma = \{\mathbf{X} \mid \det(\mathbf{X}) = 0\}$ and let

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Find the minimum distance from $\mathbf{A}$ to Σ and exhibit a specific matrix $\mathbf{S} \in \Sigma$ that achieves this minimum.

Problem 4 (2004 Putnam A3)**.** Define a sequence $\{u_n\}_{n=0}^{\infty}$ by $u_0 = u_1 = u_2 = 1$, and thereafter by the condition that

$$\det \begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} = n!$$

for all $n \geqslant 0$. Show that u_n is an integer for all n. (By convention, $0! = 1$.)

Problem 5 (Putnam 1991 A2)**.** Let $\mathbf{A}$ and $\mathbf{B}$ be different $n \times n$ matrices with real entries. If $\mathbf{A^3} = \mathbf{B^3}$ and $\mathbf{A^2 B} = \mathbf{B^2 A}$, can $\mathbf{A^2} + \mathbf{B^2}$ be invertible?

Problem 6 (Putnam 1990 A5)**.** If $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices of the same size such that $\mathbf{ABAB} = \mathbf{0}$, for which n does it follow that $\mathbf{BABA} = \mathbf{0}$?

Problem 7 (USAMO 1979)**.** Determine all non-negative integral solutions $(n_1, n_2, \ldots, n_{14})$ if any, apart from permutations, of the Diophantine Equation $n_1^4 + n_2^4 + \cdots + n_{14}^4 = 1599$.

Problem 8 (Putnam 1990 B4)**.** Let G be a finite group of order n generated by a and b. Prove or disprove: there is a sequence

$$g_1, g_2, g_3, \ldots, g_{2n}$$

such that

(1) every element of G occurs exactly twice, and
(2) g_{i+1} equals $g_i a$ or $g_i b$ for $i = 1, 2, \ldots, 2n$. (Interpret g_{2n+1} as g_1.)

Date: October 23, 2025.