WEEK 9

Problem 1 (Data science/stat interview questions)**.** For full rank $A \in \mathbb{R}^{m\times n}$ with $m > n$ (i.e., $\operatorname{rank}(A) = n$), solve

$$\min_{x\in\mathbb{R}^n} \|Ax - b\|.$$

Want another question? What is the solution of

$$\min_{x\in\mathbb{R}^n : Ax=b} \|x\|_2$$

for full rank $A \in \mathbb{R}^{m\times n}$ with $m < n$ (i.e., $\operatorname{rank}(A) = m$)?

Problem 2 (Math 1187H, 2024)**.** Suppose a weighted coin gives heads with probability α for $0 < \alpha < 1$. A game is designed as follows: flip the coin twice. If both flips are the same, repeat. If the flips are HT, A wins, and if the flips are TH, B wins. What is the expected runtime of the game?

Problem 3 (Putnam 1989 A4)**.** If α is an irrational number, $0 < \alpha < 1$, is there a finite game with an honest coin such that the probability of one player winning the game is α? (An honest coin is one for which the probability of heads and the probability of tails are both $\frac{1}{2}$. A game is finite if with probability 1 it must end in a finite number of moves.)

Problem 4 (Missouri Collegiate Competition 2017)**.** Let n be a fixed positive integer. Let $M_n(\mathbb{Z}_2)$ denote the set of $n \times n$ matrices with entries in $\mathbb{Z}_2$. Given $A \in M_n(\mathbb{Z}_2)$, what is the probability that A is invertible? (Hint: Recall that $\mathbb{Z}_2 = \{0, 1\}$ and all arithmetic is modulo 2.)

Problem 5 (AIME 1988)**.** It is possible to place positive integers into the vacant twenty-one squares of the 5 times 5 square shown below so that the numbers in each row and column form arithmetic sequences. Find the number that must occupy the vacant square marked by the asterisk (*).

			*	
	74			
				186
		103		
0				

Problem 6 (Missouri Collegiate Competition 2017)**.** Recall that the trace $\operatorname{tr}(A)$ of an $n \times n$ matrix A is the sum of the entries on the main diagonal of A. Let $A^\top$ denote the transpose of the matrix A. Show that if A and B are $n \times n$ real-valued matrices with $\operatorname{tr}(AA^\top + BB^\top) = \operatorname{tr}(AB + A^\top B^\top)$, then it must be the case that $A = B^\top$.

Problem 7 (Putnam 1991 A4)**.** Does there exist an infinite sequence of closed discs $D_1, D_2, D_3, \ldots$ in the plane, with centers $c_1, c_2, c_3, \ldots$, respectively, such that

(1) the c_i have no limit point in the finite plane,
(2) the sum of the areas of the D_i is finite, and
(3) every line in the plane intersects at least one of the D_i?

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