Homework Sheet 3: Problem 7 Solution

(7) *Consider the determinant map* $\det : \mathbb{R}^{3\times3} \to \mathbb{R}$*. Show that* $\det$ *is differentiable at each* $A \in \mathbb{R}^{3\times3}$ *in the sense that there exists a linear mapping* $\mathcal{T}_A : \mathbb{R}^{3\times3} \to \mathbb{R}$ *satisfying*

$$\lim_{\|B\|\to 0} \frac{|\det(A+B) - \det(A) - \mathcal{T}_A B|}{\|B\|} = 0,$$

where $\|B\|$ *denotes the matrix norm of* $B \in \mathbb{R}^{3\times3}$*. Compute the linear mapping* $\mathcal{T}_A$*.*

Denote by respectively A_{ij}, B_{ij} and $(A+B)_{ij}$ the submatrices in $\mathbb{R}^{2\times2}$ formed by deleting the i-th row and j-th column from $A, B, A+B$ respectively. For $1 \le j \le 3$ and $1 \le k < l \le 3$ s.t. $k, l \neq j$, by the direct computation, we have,

$$\det((A+B)_{1j}) = \det A_{1j} + \det B_{1j} + \begin{vmatrix} a_{2k} & a_{2l} \\ b_{3k} & b_{3l} \end{vmatrix} + \begin{vmatrix} b_{2k} & b_{2l} \\ a_{3k} & a_{3l} \end{vmatrix}$$

Therefore,

$$\begin{aligned} \det(A+B) - \det(A) &= \sum_{j=1}^{3} (-1)^{j+1}((a_{1j} + b_{1j}) \det(A+B)_{1j} - a_{1j} \det A_{1j}) \\ &= \sum_{j=1}^{3} (-1)^{j+1}(a_{1j}(\det B_{1j} + \begin{vmatrix} a_{2k} & a_{2l} \\ b_{3k} & b_{3l} \end{vmatrix} + \begin{vmatrix} b_{2k} & b_{2l} \\ a_{3k} & a_{3l} \end{vmatrix}) \\ &\quad + b_{1j}(\det A_{1j} + \det B_{1j} + \begin{vmatrix} a_{2k} & a_{2l} \\ b_{3k} & b_{3l} \end{vmatrix} + \begin{vmatrix} b_{2k} & b_{2l} \\ a_{3k} & a_{3l} \end{vmatrix})) \end{aligned}$$

From the definition of the Frobenius norm, we have $b_{ij}^2 \le \|B\|^2$. Therefore, $\frac{b_{ij}}{\|B\|} \le 1$. This implies that $\lim_{b_{ij}, b_{kl} \to 0} b_{kl} \frac{b_{ij}}{\|B\|} = 0$. Therefore $\det B_{1j}$, $b_{1j} \begin{vmatrix} a_{2k} & a_{2l} \\ b_{3k} & b_{3l} \end{vmatrix}$, and $b_{1j} \begin{vmatrix} b_{2k} & b_{2l} \\ a_{3k} & a_{3l} \end{vmatrix}$ vanish when divided by $\|B\| \to 0$.

Thus,

$$\tau_A B = \sum_{j=1}^{3} a_{1j} \begin{vmatrix} a_{2k} & a_{2l} \\ b_{3k} & b_{3l} \end{vmatrix} + a_{1j} \begin{vmatrix} b_{2k} & b_{2l} \\ a_{3k} & a_{3l} \end{vmatrix} + b_{1j} \det A_{1j}$$

$$\begin{aligned}
= \quad & a_{11} \begin{vmatrix} a_{22} & a_{23} \\ b_{32} & b_{33} \end{vmatrix} + a_{11} \begin{vmatrix} b_{22} & b_{23} \\ a_{32} & a_{33} \end{vmatrix} + b_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \\
& - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ b_{31} & b_{33} \end{vmatrix} - a_{12} \begin{vmatrix} b_{21} & b_{23} \\ a_{31} & a_{33} \end{vmatrix} - b_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \\
& + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ b_{31} & b_{32} \end{vmatrix} + a_{13} \begin{vmatrix} b_{21} & b_{22} \\ a_{31} & a_{32} \end{vmatrix} + b_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\
= \quad & b_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - b_{21}(a_{12}a_{33} - a_{13}a_{32}) + b_{31}(a_{12}a_{23} - a_{13}a_{22}) \\
& - b_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + b_{22}(a_{11}a_{33} - a_{13}a_{31}) - b_{32}(a_{11}a_{23} - a_{13}a_{21}) \\
& + b_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} - b_{23}(a_{11}a_{32} - a_{12}a_{31}) + b_{33}(a_{11}a_{22} - a_{12}a_{21}) \\
= \quad & b_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - b_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + b_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\
& - b_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + b_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - b_{32} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\
& + b_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} - b_{23} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} + b_{33} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \\
& = \text{trace (Adj (A) B)}
\end{aligned}$$

where Adj (A) is the transpose of cofactor matrix of A.