Homework Sheet 5: Solutions

1. *Consider the non-empty set of functions*

$$V := \left\{ p : \mathbb{R} \to \mathbb{R} \,|\, p(x) = \sum_{k=0}^{n-1} a_k x^k \text{ for } a_i \in \mathbb{R}, \quad x \in \mathbb{R} \right\}$$

   (a) *Define an addition operation $+ : V \times V \to V$ and a scalar multiplication operation $\cdot : \mathbb{R} \times V \to V$ such that the triple $(V, +, \cdot)$ is a real vector space.*

   We define $+ : V^2 \to V$ by $(p_1 + p_2)(x) = p_1(x) + p_2(x)$. Then $p_1(x) + p_2(x) = \sum_{k=0}^{n-1}(a_{1k} + a_{2k})x^k$ where $p_1(x) = \sum_{k=0}^{n-1} a_{1k}x^k$ and $p_2(x) = \sum_{k=0}^{n-1} a_{2k}x^k$. Therefore, $p_1 + p_2 \in V$. This confirms that V is closed under addition.

   Similarly, we define $\cdot : \mathbb{R} \times V \to V$ by $(\lambda p)(x) = \lambda p(x)$. Then $\lambda p(x) = \sum_{k=0}^{n}(\lambda a_k)x^k$ where $p(x) = \sum_{k=0}^{n} a_k x^k$, so $\lambda p \in V$. This confirms that V is closed under scalar multiplication.

   [V1] $(p_1 + p_2)(x) = \sum_{k=0}^{n-1}(a_{1k} + a_{2k})x^k = \sum_{k=0}^{n-1}(a_{2k} + a_{1k})x^k = (p_2 + p_1)(x)$ Thus $p_1 + p_2 = p_2 + p_1$ for any $p_1, p_2 \in V$.

   [V2] For p_3 defined by $p_3(x) = \sum_{k=0}^{n-1} a_{3k}x^k$, we have

$$\begin{aligned}(p_1 + (p_2 + p_3))(x) =& \sum_{k=0}^{n-1} a_{1k}x^k + (\sum_{k=0}^{n-1} a_{2k}x^k + \sum_{k=0}^{n-1} a_{3k}x^k) \\ =& (\sum_{k=0}^{n-1} a_{1k}x^k + \sum_{k=0}^{n-1} a_{2k}x^k) + \sum_{k=0}^{n-1} a_{3k}x^k \\ =& ((p_1 + p_2) + p_3)(x)\end{aligned}$$

   Thus $p_1 + (p_2 + p_3) = (p_1 + p_2) + p_3$ for any $p_1, p_2, p_3 \in V$.

   [V3] Define 0 by $0(x) = 0$. By the Fundamental Theorem of Algebra, an $n - 1$ degree polynomial with more than $n - 1$ roots is identically zero. Therefore $0 \in V$ is uniquely defined by $0(x) = \sum_{k=0}^{n-1} 0 \cdot x^k$, and we have

$$(p+0)(x) = \sum_{k=0}^{n-1} a_k x^k + \sum_{k=0}^{n-1} 0 \cdot x^k = p(x)$$

Therefore, $p + 0 = p$ for any $p \in V$.

[V4] We have

$$(p + (-1) \cdot p)(x) = \sum_{k=0}^{n-1} a_k x^k + (-1) \sum_{k=0}^{n-1} a_k \cdot x^k = 0$$

Therefore, $p + (-1)p = 0$ for any $p \in V$.

[V5] We have

$$(1 \cdot p)(x) = 1 \cdot \sum_{k=0}^{n-1} a_k x^k = p(x)$$

Therefore, $1 \cdot p = p$ for any $p \in V$.

[V6] We have

$$c_1(c_2 \cdot p)(x) = c_1(c_2 \cdot \sum_{k=0}^{n-1} a_k x^k) = (c_1 c_2) \cdot \sum_{k=0}^{n-1} a_k x^k = (c_1 c_2) p(x)$$

Therefore, $c_1(c_2 \cdot p) = (c_1 c_2) p$ for any $p \in V$.

[V7] We have

$$((c_1 + c_2)p)(x) = (c_1 + c_2) \cdot \sum_{k=0}^{n-1} a_k x^k = c_1 \sum_{k=0}^{n-1} a_k x^k + c_2 \sum_{k=0}^{n-1} a_k x^k = c_1 p(x) + c_2 p(x)$$

Therefore, $(c_1 + c_2) \cdot p = c_1 p + c_2 p$ for any $c_1, c_2 \in \mathbb{R}$ and $p \in V$.

[V8] We have

$$\begin{aligned} c(p_1 + p_2)(x) &= c(\sum_{k=0}^{n-1} a_{1k} x^k + \sum_{k=0}^{n-1} a_{2k} x^k) \\ &= c \sum_{k=0}^{n-1} a_{1k} x^k + c \sum_{k=0}^{n-1} a_{2k} x^k = c p_1(x) + c p_2(x) \end{aligned}$$

Therefore, $c \cdot (p_1 + p_2)p = c_1 \cdot p_1 + c \cdot p_2$ for any $c \in \mathbb{R}$ and $p_1, p_2 \in V$.

(b) *Find a basis for this vector space, and deduce its dimension.*

By the construction of V, the set of monomials $b = \{x^k\}_{k=0}^{n-1}$ is a spanning set. By the Fundamental Theorem of Algebra, an $n-1$ degree polynomial with more than $n-1$ roots is identically zero. Therefore $\sum_{k=0}^{n-1} a_k \cdot x^k = 0$ for all $x \in \mathbb{R}$ if and only if $a_k = 0$ for $0 \leq k \leq n-1$. Thus, b is a basis of V.

2. *Suppose $m, n \geq 1$ are integers.*

(a) *Prove that the set of all maps $f : \mathbb{R}^m \to \mathbb{R}^n$ of class $C^1(\mathbb{R}^m)$ admits the structure of a real vector space with respect to the 'natural' $+ : C^1(\mathbb{R}^m) \times C^1(\mathbb{R}^m) \to C^1(\mathbb{R}^m)$ and $\cdot : \mathbb{R} \times C^1(\mathbb{R}^m) \to C^1(\mathbb{R}^m)$ operations.*

We define $+ : C^1(\mathbb{R}^m) \times C^1(\mathbb{R}^m) \to C^1(\mathbb{R}^m)$ by $(f+g)(x) = f(x) + g(x)$. Then, the fact that f and g are differentiable implies that there exists $f', g' : \mathbb{R}^m \to \mathbb{R}^{n \times m}$ such that for every $a \in \mathbb{R}^m$, we have

$$\lim_{|h| \to 0} \frac{f(a+h) - f(a) - f'(a)}{|h|} = \lim_{|h| \to 0} \frac{g(a+h) - g(a) - g'(a)}{|h|} = 0$$

Consequently,

$$\lim_{|h| \to 0} \frac{(f(a+h) + g(a+h)) - (f(a) + g(a)) - (f'(a) + g'(a))}{|h|}$$
$$= \lim_{|h| \to 0} \frac{(f+g)(a+h) - (f+g)(a) - (f'(a) + g'(a))}{|h|} = 0$$

Therefore, there exists $(f+g)'$ on $\mathbb{R}^m$ defined by $(f+g)'(a) = f'(a) + g'(a)$. The existence of $(f+g)'$ implies that $(f+g)'$ is continuous on $\mathbb{R}^m$ (Munkres, Theorem 5.2).

Next, let $D_j(f+g)$ be the j-th partial derivative of $(f+g)$, which by the preceding paragraph is defined by $D_j(f+g) = D_j f + D_j g$. The continuity of $D_j f + D_j g$ implies that for every $\epsilon/2 > 0$, there exists $\delta = \min(\delta_f, \delta_g)$, such that

$$\begin{aligned}\|D_j(f+g)(x) - D_j(f+g)(y)\| &= \|D_j f(x) - D_j f(y)\| + \|D_j g(x) - D_j g(y)\| \\ &\leq \epsilon/2 + \epsilon/2\end{aligned}$$

for $\|x - y\| \leq \delta$. This confirms that $(f+g) \in C^1(\mathbb{R}^m)$, and therefore $C^1(\mathbb{R}^m)$ is closed under addition.

Similarly, we define $\cdot : \mathbb{R} \times C^1(\mathbb{R}^m) \to C^1(\mathbb{R}^m)$by $(\lambda f)(x) = \lambda f(x)$. Then, the fact that f is differentiable implies that there exists $f' : \mathbb{R}^m \to \mathbb{R}^{n \times m}$ such that for every $a \in \mathbb{R}^m$, we have

$$\lim_{|h| \to 0} \frac{f(a+h) - f(a) - f'(a)}{|h|} = 0$$

Consequently,

$$\lim_{|h| \to 0} \frac{\lambda f(a+h) - \lambda f(a) - \lambda f'(a)}{|h|} = \lim_{|h| \to 0} \frac{(\lambda f)(a+h) - (\lambda f)(a) - \lambda f'(a)}{|h|} = 0$$

Therefore, there exists $(\lambda f)'$ on $\mathbb{R}^m$ defined by $(\lambda f)'(a) = \lambda f'(a)$. The existence of $\lambda f'$ implies that f' is continuous on $\mathbb{R}^m$.

Next, let $D_j(\lambda f)$ be the j-th partial derivative of f, which by the preceding paragraph is defined by $D_j(\lambda f) = D_j \lambda f$. The continuity of $D_j f$ implies that $D_j \lambda f$ is also continuous (Munkres, Theorem 3.6). This confirms that $(\lambda f) \in C^1(\mathbb{R}^m)$, and therefore $C^1(\mathbb{R}^m)$ is closed under scalar multiplication.

[V1] We have

$$(f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x)$$

Thus $f + g = g + f$ for any $f, g \in V$.

[V2] For $h \in C^1(\mathbb{R}^m)$, we have

$$\begin{aligned}(f + (g + h))(x) =& f(x) + (g(x) + h(x)) \\ =& (f(x) + g(x)) + h(x) = ((f + g) + h)(x)\end{aligned}$$

Thus $f + (g + h) = (f + g) + h$ for any $f, g, h \in C^1(\mathbb{R}^m)$.

[V3] Define 0 by $0(x) = 0$. Trivially $0' = 0$, and $D_j 0 = 0$, and therefore $0 \in C^1(\mathbb{R}^m)$. We have,

$$(f + 0)(x) = f(x) + 0 = f(x)$$

Therefore, $f + 0 = f$ for any $f \in V$. Note that if $0(x) \neq 0$ for any x, then the above equality will not hold, and therefore, $0 \in C^1(\mathbb{R}^m)$ is uniquely defined.

[V4] We have

$$(f + (-1) \cdot f)(x) = f(x) + (-1)f(x) = 0$$

Therefore, $f + (-1)f = 0$ for any $f \in C^1(\mathbb{R}^m)$.

[V5] We have

$$(1 \cdot f)(x) = f(x)$$

Therefore, $1 \cdot f = f$ for any $f \in C^1(\mathbb{R}^m)$.

[V6] We have

$$c_1(c_2 \cdot f)(x) = c_1(c_2 f(x)) = (c_1 c_2) \cdot f(x) = (c_1 c_2) f(x)$$

Therefore, $c_1(c_2 \cdot f) = (c_1 c_2) f$ for any $f \in C^1(\mathbb{R}^m)$.

[V7] We have

$$((c_1 + c_2) f)(x) = (c_1 + c_2) \cdot f(x) = c_1 f(x) + c_2 f(x) = c_1 f(x) + c_2 f(x)$$

Therefore, $(c_1 + c_2) \cdot f = c_1 f + c_2 f$ for any $c_1, c_2 \in \mathbb{R}$ and $f \in C^1(\mathbb{R}^m)$.

[V8] We have

$$c(f_1 + f_2)(x) = c(f_1(x) + f_2(x)) = c f_1(x) + c f_2(x)$$

Therefore, $c \cdot (f_1 + f_2) p = c_1 \cdot f_1 + c \cdot f_2$ for any $c \in \mathbb{R}$ and $f_1, f_2 \in C^1(\mathbb{R}^m)$.

(b) *In the case $n = m = 1$, show that this vector space cannot be finite-dimensional.*

Assume that $\{b_i\}_{i=0}^k$ is a basis of $C^1(\mathbb{R})$. Since $(x^i)' = ix^{i-1}$, each monomial is of class $C^1(\mathbb{R})$, and as shown above, the set of monomials is linearly independent. We can have l linearly independent monomials for any $l > k$. This contradicts Theorem 1.1 in Munkres.

3. *Let $V := \mathbb{R}^3$ and consider the set of all 3-tensors $\mathcal{L}^3(V)$. Give two examples, say f and g, of maps which lie in $\mathcal{L}^3(V)$.*

   Consider $f(x, y, z) = x_1y_1z_1$ and $g(x, y, z) = x_1y_1z_1 + x_2y_2z_2$ where $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3), z = (z_1, z_2, z_3) \in \mathbb{R}^3$.

   For $s = (s_1, s_2, s_3)$, consider $f(\alpha x + \beta s, y, z) = (\alpha x_1 + \beta s_1)y_1z_1 = \alpha x_1y_1z_1 + \beta s_1y_1z_1 = \alpha f(x, y, z)+\beta f(s, y, z)$ and $g(\alpha x+\beta s, y, z) = (\alpha x_1+\beta s_1)y_1z_1+(\alpha x_2+\beta s_2)y_2z_2 = \alpha(x_1y_1z_1 + x_2y_2z_2) + \beta(s_1y_1z_1 + s_2y_2z_2) = \alpha g(x, yz) + \beta g(s, y, z)$. We can similarly confirm the linearity in the second and third variables.

4. *Suppose $f_1, ..., f_k \in \mathcal{L}^1(V)$, the set of all 1-tensors on a real vector space V. Prove that*

$$F(x_1, ..., x_k) := f_1(x_1)...f_k(x_k) \quad \text{for } (x_1, ..., x_k) \in V^k$$

   is a k-tensor on V, i.e. $F \in \mathcal{L}^k(V)$.

   For $s \in V$, by linearity of f_1 we have $F(\alpha x_1 + \beta s, x_2, ..., x_k) = f_1(\alpha x_1 + \beta s)f_2(x_2)...f_k(x_k) = \alpha f_1(x_1)f_2(x_2)...f_k(x_k)+\beta f_1(s)f_2(x_2)...f_k(x_k) = \alpha F(x_1, ..., x_k)+ \beta F(s, ..., x_k)$. We can similarly confirm the linearity in the other variables.

5. *Let V be a real vector space, and $k \geq 1$ an integer. Show that if $f, g \in \mathcal{L}^k(V)$ and $c, d \in \mathbb{R}$, then $cf + dg \in \mathcal{L}^k(V)$.*

   For $s \in V$, by multilinearity of f and g, we have

$$\begin{aligned}
(cf + dg)(\alpha x_1 + \beta s, x_2, ..., x_k) &= cf(\alpha x_1 + \beta s, x_2, ..., x_k) + dg(\alpha x_1 + \beta s, x_2, ..., x_k) \\
&= c(\alpha f(x_1, x_2, ..., x_k) + \beta f(s, x_2, ..., x_k)) + d(\alpha g(x_1, x_2, ..., x_k) + \beta g(s, x_2, ..., x_k)) \\
&= \alpha[cf(x_1, x_2, ..., x_k) + dg(x_1, x_2, ..., x_k)] + \beta[cf(s, x_2, ..., x_k)) + dg(s, x_2, ..., x_k)] \\
&= \alpha(cf + dg)(x_1, x_2, ..., x_k) + \beta(cf + dg)(s, x_2, ..., x_k)
\end{aligned}$$

   We can similarly confirm the linearity in the other variables.

6. *Let $V := \mathbb{R}^4$. Which of the following define 2-tensors on V?*

(a) $f(x, y) := 3x_1y_2 + 5x_2x_3$;

We have $f((0, 1, 1, 0) + (0, 1, 1, 0), 0) = 20$ and $f((0, 1, 1, 0), 0) + f((0, 1, 1, 0), 0) = 10$. Thus $f \notin \mathcal{L}^2(V)$.

(b) $g(x, y) := x_1y_2 + x_2y_4 + 1$;

We have $g(0 + 0, 0) = 1$ and $g(0, 0) + g(0, 0) = 2$. Thus $g \notin \mathcal{L}^2(V)$.

(c) $h(x, y) := x_1y_1 - 7x_2y_3$.

$$\begin{aligned} h(\alpha x + \beta s, y) =& (\alpha x_1 + \beta s_1)y_1 - 7(\alpha x_2 + \beta s_2)y_3 \\ =& \alpha(x_1y_1 - 7x_2y_3) + \beta(s_1y_1 - 7s_2y_3) \\ =& \alpha h(x, y) + \beta h(s, y) \end{aligned}$$

Similarly,

$$\begin{aligned} h(x, \alpha y + \beta s) =& x_1(\alpha y_1 + \beta s_1) - 7x_2(\alpha y_3 + \beta s_3) \\ =& \alpha(x_1y_1 - 7x_2y_3) + \beta(x_1y_1 - 7x_2s_3) \\ =& \alpha h(x, y) + \beta h(x, s) \end{aligned}$$

Thus $h \in \mathcal{L}^2(V)$.

7. *Let V be a real vector space with basis $\{v_k\}_{k=1}^n$. To begin, for each $1 \le i \le n$ we define the maps $\phi_i : \{v_k\}_{k=1}^n \to \{0, 1\}$ by*

$$\phi_i(v_j) := \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } i \neq j. \end{cases}$$

(a) *How does ϕ_i extend in a natural way to a linear map $\tilde{\phi}_i : V \to \mathbb{R}$ on the whole vector space V? Write out $\tilde{\phi}_i : V \to \mathbb{R}$ explicitly (that is, write out $\tilde{\phi}_i(x)$ for any $x \in V$).*

Define $\tilde{\phi}_i : V \to \mathbb{R}$ by $\tilde{\phi}_i(v) = a_i$ where $v = \sum_{k=1}^n a_kv_k$. Observe that $\tilde{\phi}_i = \phi_i$ on the restricted set $\{v_k\}_{k=1}^n$.

(b) *How can we use Proposition 8.1 to show that ϕ_i extends to a linear map on the whole vector space V uniquely?*

For $w = \sum_{k=1}^{n} b_k v_k$, we have $\tilde{\phi}_i(\alpha v + \beta w) = \alpha a_i + \beta b_i = \alpha \tilde{\phi}_i(v) + \beta \tilde{\phi}_i(w)$. This confirms that $\tilde{\phi}_i \in \mathcal{L}^1(V)$. This and 7(a) establish the hypothesis of proposition 8.1. Therefore, given any $\tilde{\phi}_i' \in \mathcal{L}^1(V)$ such that $\tilde{\phi}_i' = \phi_i = \tilde{\phi}_i$ on the restricted set $\{v_k\}_{k=1}^{n}$, we have $\tilde{\phi}_i' = \tilde{\phi}_i$ on the whole vector space V, i.e. the extension given by $\tilde{\phi}_i$ is unique.

(c) *Suppose $V := \mathbb{R}^3$ equipped with the canonical basis $\{e_k\}_{k=1}^{3}$. Can you give a 'geometric' description of the map $\phi_1 : V \to \mathbb{R}$ associated with e_1?*

ϕ_1 is a scalar projection of $v \in V$ on e_1, i.e. $\phi_1(v) = \langle e_1, v \rangle$.

8. *Let V be a real vector space with basis $\{v_r\}_{r=1}^{n}$, and $k \geq 1$ an integer. For any fixed multi-index $I = (i(1), ..., i(k)) \in \{1, ..., n\}^k$, we define an associated map $\phi_I : \underbrace{\{v_r\}_{r=1}^{n} \times ... \times \{v_r\}_{r=1}^{n}}_{k \text{ times}} \to \{0, 1\}$ by*

$$\phi_I(v_{j(1)}, ..., v_{j(k)}) := \begin{cases} 1 & \text{if } (j(1), ..., j(k)) = I, \\ 0 & \text{if } (j(1), ..., j(k)) \neq I. \end{cases} \tag{1}$$

(a) *How does ϕ_I extend in a natural way to a multilinear map $\tilde{\phi}_I : V^k \to \mathbb{R}$ on the whole vector space V? Write out $\tilde{\phi}_I : V^k \to \mathbb{R}$ explicitly (that is, write out $\tilde{\phi}_I(x_1, ..., x_k)$ for any $(x_1, ..., x_k) \in V^k$).*

Define $\tilde{\phi}_I : V \to \mathbb{R}$ by $\tilde{\phi}_I(x_1, ..., x_k) = \prod_{m=1}^{k} x_{m,i(m)}$ where $x_m = \sum_{r=1}^{n} x_{m,r} v_r$. Observe that $\tilde{\phi}_I = \phi_I$ on the restricted set $\underbrace{\{v_r\}_{r=1}^{n} \times ... \times \{v_r\}_{r=1}^{n}}_{k \text{ times}}$.

(b) *How can we use proposition 8.1. to show that $\tilde{\phi}_I$ extends to a map on the whole vector space V^k uniquely?*

For $y = \sum_{r=1}^{n} y_r v_r$, we have

$$\begin{aligned} \tilde{\phi}_I(\alpha x_1 + \beta y, x_2, ..., x_k) =& (\alpha x_{1,i(1)} + \beta y_{i(1)}) \prod_{m=2}^{k} x_{m,i(m)} \\ =& \alpha \tilde{\phi}_I(x_1, x_2, ..., x_k) + \beta \tilde{\phi}_I(y, x_2, ..., x_k) \end{aligned}$$

This confirms that $\tilde{\phi}_I \in \mathcal{L}^k(V)$. This and 8(a) establish the hypothesis of proposition 8.1. Therefore, given any other $\tilde{\phi}_I' \in \mathcal{L}^k(V)$ such that $\tilde{\phi}_I' = \phi_I = \tilde{\phi}_I$ on the restricted set $\underbrace{\{v_r\}_{r=1}^{n} \times ... \times \{v_r\}_{r=1}^{n}}_{k \text{ times}}$, we have $\tilde{\phi}_I' = \tilde{\phi}_I$ on the whole vector space V^k, i.e., the extension given by $\tilde{\phi}_I$ is unique.

9. *Let V be a real vector space and $k \geq 1$ an integer. Prove that the set of n^k k-tensors*

$$\{\phi_I \;:\; I \in \{1, ...n\}^k\} \subset \mathcal{L}^k(V)$$

defined in question 8 above is a basis for $\mathcal{L}^k(V)$.

See Munkres, Theorem 26.3.