Quiz Sheet 1: Solutions

Question 1. (Exercise from Lecture 1.) *Show that Green's Theorem is a special case of Stokes' Theorem.*

$$\begin{aligned}
\int_{\partial S} P(x,y)dx + Q(x,y)dy &= \int_{\partial S} (P(x,y), Q(x,y), 0) \cdot (dx, dy, dz) \\
&= \int_S \nabla \times (P(x,y), Q(x,y), 0) \cdot dS = \\
&= \int_S \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} \cdot dS \\
&= \int_S (\frac{\partial}{\partial x} Q(x,y) - \frac{\partial}{\partial x} P(x,y))k \cdot k \, dxdy
\end{aligned}$$

where S is the region of the xy plane bounded by ∂S which is positively oriented with respect to $k = (0,0,1)$, the normal unit vector in the direction of the z-axis.

Question 2. (Exercise 1 from Lecture 2.) *Suppose $U \subseteq \mathbb{R}^m$ is open. Show that $f : U \to \mathbb{R}^n$ is continuous on U iff for every open set $V \subseteq \mathbb{R}^n$, the set*

$$f^{-1}(V) := \{x \in U : f(x) \in V\}$$

is open in $\mathbb{R}^m$.

($\Rightarrow$) Take any open set $V \subseteq \mathbb{R}^n$, and any arbitrary point $x \in f^{-1}(V)$. Since V is open, and $f(x) \in V$, by continuity of f, there exists an open set $O \subseteq U$ containing x s.t. $f(O) \subseteq V$. This implies $O \subseteq f^{-1}(V)$. Thus O is a neighborhood of x. Therefore, there exists $\epsilon > 0$ s.t. $B(x, \epsilon) \subseteq O \subseteq f^{-1}(V)$. Since x is an arbitrary point in $f^{-1}(V)$, this implies that $f^{-1}(V)$ is open.

($\Leftarrow$) Take any point $x \in U$ and any open set $V \subseteq \mathbb{R}^n$ s.t. $f(x) \in V$. The hypothesis implies that $f^{-1}(V)$ is open. Furthermore, since $f(x) \in V$, we have $x \in f^{-1}(V) \subseteq \mathbb{R}^m$. Lastly $f(f^{-1}(V)) = V$. Thus we found an open set $f^{-1}(V) \subset R^m$ containing x such that $f(f^{-1}(V)) = V$. This confirms that f is continuous at x. Since x is an arbitrary point in U, f is continuous on U.

Question 3. (Exercise 2 from Lecture 2.) *Suppose $U \subseteq \mathbb{R}^m$ is open. Show that $f : U \to \mathbb{R}^n$ is continuous at $x_0 \in U$ iff for every $\epsilon > 0$, there exists $\delta > 0$ s.t.*

$$d(x_0, x) < \delta \Rightarrow d(f(x_0), f(x)) < \epsilon \ (*)$$

($\Rightarrow$) Consider $B(f(x_0), \epsilon) \in \mathbb{R}^n$. Since it is an open set containing $f(x_0)$, by continuity of f, there exists an open set $O \subseteq \mathbb{R}^m$ containing x_0, s.t. $f(O) \subseteq B(f(x_0), \epsilon)$. By the definition of an open set, there exists $\delta > 0$, s.t. $B(x_0, \delta) \subseteq O$. Thus, if $x \in B(x_0, \delta)$, this implies that $f(x) \in f(B(x_0, \delta)) \subseteq f(O) \subseteq B(f(x_0), \epsilon)$, which is equivalent to $(*)$.

($\Leftarrow$) Let x_0 be any point in U and $V \subseteq \mathbb{R}^n$ be an open set containing $f(x_0)$. Then, by the definition of the open set, there exists $\epsilon > 0$ s.t. $B(f(x_0), \epsilon) \subseteq V$. By the hypothesis, there exists $\delta > 0$ s.t. $(*)$ holds. Therefore, we found an open set $B(x_0, \delta)$ containing x_0, and

$$f(B(x_0, x)) \subseteq B(f(x_0), \epsilon) \subseteq V$$

Therefore, f is continuous at x_0.

Question 4. (Exercise 3 from Lecture 2) *Show that $x_0 := (1, 0, ..., 0)$, where there are $m - 1$ zero entries, is a limit point of the open unit ball*

$$A = \{x \in \mathbb{R}^m : \|x\| < 1\}$$

Consider $x_\epsilon = (1-\epsilon/2, 0, ..., 0) \neq x_0$. (If $\epsilon > 1$, we can take $x_\epsilon = (1/2, 0, ..., 0)$). Thus, for every $\epsilon > 0$, we have

$$\|x_\epsilon - x_0\| \leq \epsilon/2 \Rightarrow x_\epsilon \in B(x_0, \epsilon)$$

and

$$\|x_\epsilon - 0\| < 1 \Rightarrow x_\epsilon \in A$$

which implies the desired result.