Quiz Sheet 2

Instructions: Turn in responses to TWO questions of your choice. This quiz will be timed for 15 minutes.

Question 1. Do all of the following:

(a) Show that $f : \mathbb{R}^m \to \mathbb{R}^n$ having the form:

$$f(x) = (f_1(x), ..., f_n(x))$$

is continuous at $x_0 \in \mathbb{R}^m$ if and only if each *component function* $f_i : \mathbb{R}^m \to \mathbb{R}$ is continuous at x_0.

Hint: Use the result we established previously: Suppose $U \subseteq \mathbb{R}^m$ is open. $f : U \to \mathbb{R}^n$ is continuous at $x_0 \in U$ iff for every $\epsilon > 0$, there exists $\delta > 0$ s.t. $d(x_0, x) < \delta \Rightarrow d(f(x_0), f(x)) < \epsilon$

To show the "if" part, ensure that $|f_i(x) - f_i(x_0)| < \frac{\epsilon}{\sqrt{n}}$.

(b) Show that $g : \mathbb{R}^m \to \mathbb{R}^m$ given by $g(x) := ||x||^2 x$ is continuous on $\mathbb{R}^m$.

Hint: Use (a) above, the standard result in $\mathbb{R}$ that if f and g are continous mappings from a metric space X to $\mathbb{R}$, then fg is a continuous mapping from X to $\mathbb{R}$, and the result shown in class that $f : \mathbb{R}^m \to \mathbb{R}$ given by $f(x) := ||x||$ is continuous on $\mathbb{R}^m$.

Question 2. Do all of the following:

(a) Suppose $A \subseteq \mathbb{R}^m$, and let $f : A \to \mathbb{R}^n$. State what it means for *$f(x)$ to approach y_0 as x approaches x_0*, i.e., $f(x) \to y_0$ as $x \to x_0$ or $\lim_{x \to x_0} f(x) = y_0$.

(b) Show that $f(x)$ approaches y_0 as x approaches x_0 iff for every $\epsilon > 0$ there exists $\delta > 0$ s.t.

$$x \in A, 0 < d(x_0, x) < \delta \Rightarrow d(y_0, f(x)) < \epsilon$$

Question 3. Do all of the following:

(a) Let $A \subseteq \mathbb{R}^m$ and $\varphi : A \to \mathbb{R}^n$. Suppose that A contains a neighborhood of $a \in A$. State the definition of what it means for φ to be *differentiable* at a as well as the definition of the *derivative of φ at a*, denoted as $D\varphi(a)$.

(c) Find $D\varphi(a)$ for any $a \in \mathbb{R}^m$ if $\varphi(x) := ||x||^2$, $x \in \mathbb{R}^m$,

Question 4. Do all of the following:

(a) State what it means for $x \otimes y$ to be *the tensor product* of x and y in $\mathbb{R}^m$.

(ii) Find $D\varphi(a)$ for any $a \in \mathbb{R}^m$ if $\varphi(x) := ||x||^2 x$, $x \in \mathbb{R}^m$.