Quiz 2: Solutions

Question 1. (Exercise 3 from Lecture 2)

(a) *Show that* $f : \mathbb{R}^m \to \mathbb{R}^n$ *having the form:*

$$f(x) = (f_1(x), ..., f_n(x))$$

is continuous at $x_0 \in \mathbb{R}^m$ *if and only if each component function* $f_i : \mathbb{R}^m \to \mathbb{R}$ *is continuous at* x_0.

($\Rightarrow$) By a result we showed previously, the continuity of f at x_0 implies that for every $\epsilon > 0$, there exists $\delta > 0$ s.t.

$$\|x_0 - x\| < \delta \Rightarrow \sqrt{\sum_{i=1}^{n} |f_i(x_0) - f_i(x)|^2} = \|f(x_0) - f(x)\| < \epsilon$$

which implies that $|f_i(x_0) - f_i(x)| < \epsilon$ for every $1 \leq i \leq n$. Thus, each f_i is continuous.

($\Leftarrow$) The continuity of f_i implies that every $\epsilon > 0$, there exists $\delta > 0$ s.t. $|f_i(x_0) - f_i(x)| < \frac{\epsilon}{\sqrt{n}}$. This implies that

$$\sqrt{\sum_{i=1}^{n} |f_i(x_0) - f_i(x)|^2} = \|f(x_0) - f(x)\| < \epsilon$$

and consequently f is continuous.

(b) *Show that* $g : \mathbb{R}^m \to \mathbb{R}^m$ *given by* $g(x) := \|x\|^2 x$ *is continuous on* $\mathbb{R}^m$.

The standard result is that if f and g are continous mappings from a metric space X to $\mathbb{R}$, then fg is a continuous mapping from X to $\mathbb{R}$ and the result shown in class that $f : \mathbb{R}^m \to \mathbb{R}$ given by $f(x) := \|x\|$ is continuous on $\mathbb{R}^m$ imply that $g_i = \|x\|^2 x_i$ is continuous for all $0 \leq i \leq m$. By (a), this implies that g is continuous.

Question 2. (Exercise 5 from Lecture 2) *Show that* $f(x) \to y_0$ *as* $x \to x_0$ *iff for every* $\epsilon > 0$ *there exists* $\delta > 0$ *s.t.*

$$x \in A, 0 < d(x_0, x) < \delta \Rightarrow d(y_0, f(x)) < \epsilon$$

($\Rightarrow$) Let $B(y_0, \epsilon) \supseteq \{f(x) : d(y_0, f(x))\}$. Since $B(y_0, \epsilon) \subseteq \mathbb{R}^n$ is an open set containing y_0, the existence of the limit implies that there exists an open set $U \in \mathbb{R}^m$ containing x_0 s.t.

$$x \in U \cap A \Rightarrow f(x) \in B(y_0, \epsilon) \ (*)$$

Since U is an open set containing x_0, there exists $\delta > 0$ s.t.

$$x \in B(x_0, \delta) \subseteq U$$

By construction $x \in A$. Therefore, $x \in B(x_0, \delta) \cap A \subseteq U \cap A$. Then $(*)$ implies that $f(x) \in B(y_0, \epsilon)$.

($\Leftarrow$) Consider any open set $V \subseteq \mathbb{R}^n$ containing y_0. By the definition of an open set, there exists $\epsilon > 0$, s.t. $B(y_0, \epsilon) \subseteq V$. Then by the hypothesis, there exists $\delta > 0$ s.t.

$$x \in A, x \in B(x_0, \delta) \Rightarrow f(x) \in B(y_0, \epsilon)$$

Thus, for any open set $V \subseteq \mathbb{R}^n$ containing y_0, we have an open set $B(x_0, \delta) \in \mathbb{R}^m$ containing x_0, such that

$$x \in B(x_0, \delta) \cap A \Rightarrow f(x) \in B(y_0, \epsilon) \subseteq V$$

This confirms that $f(x) \to y_0$ as $x \to x_0$.

Question 3. (Last Exercise from Lecture 2, part (a)) *Find $\varphi'(a)$ for any $a \in \mathbb{R}^m$ if $\varphi(x) := ||x||^2, x \in \mathbb{R}^m$.*

$$\begin{aligned}\varphi(a+h) - \varphi(a) =& ||a+h||^2 - ||a||^2 \\ =& \langle a+h, a+h\rangle - \langle a, a\rangle \\ =& 2\langle a, h\rangle + ||h||^2\end{aligned}$$

Thus,

$$\lim_{||h|| \to 0} \frac{\varphi(a+h) - \varphi(a) - 2\langle a, h\rangle}{||h||} = \lim_{||h|| \to 0} ||h|| = 0$$

which shows that $\varphi'(a) = 2a$ (and thus $\varphi'(a)h = 2\langle a, h\rangle$).

Question 4. (Last Exercise from Lecture 2, part (b)) *Find* $\varphi'(a)$ *for any* $a \in \mathbb{R}^m$ *if* $\varphi(x) := \|x\|^2 x, x \in \mathbb{R}^m$.

$$\begin{aligned}\varphi(a+h)-\varphi(a) =&\|a+h\|^2(a+h)-\|a\|^2 a\\ =&\langle a+h, a+h\rangle(a+h)-\|a\|^2 a\\ =&\langle a+h, a+h\rangle a+\langle a+h, a+h\rangle h-\|a\|^2 a\\ =&(\|a\|^2+2\langle a, h\rangle+\|h\|^2)a+(\|a\|^2+2\langle a, h\rangle+\|h\|^2)h-\|a\|^2 a\\ =&2\langle a, h\rangle a+\|h\|^2 a+\|a\|^2 h+2\langle a, h\rangle h+\|h\|^2 h\end{aligned}$$

Thus,

$$\varphi(a+h)-\varphi(a)-2\langle a, h\rangle a-\|a\|^2 h=\|h\|^2 a+2\langle a, h\rangle h+\|h\|^2 h$$

and we have

$$\lim_{\|h\|\to 0}\frac{\varphi(a+h)-\varphi(a)-B(a)h}{\|h\|}=J(h)$$

where

$$B(a)h=2\langle a, h\rangle a+\|a\|^2 h=(2a\otimes a+\|a\|^2 I)h$$

and

$$J(h)=\frac{\|h\|^2 a+2\langle a, h\rangle h+\|h\|^2 h}{\|h\|}$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned}\|J(h)\| \leq&\|h\|\cdot\|a\|+2\langle a, h\rangle+\|h\|^2\\ \leq&\|h\|\cdot\|a\|+2\|a\|\cdot\|h\|+\|h\|^2 \to 0\end{aligned}$$

as $h \to 0$.

Therefore, $\varphi'(a)=B(a)=(2a\otimes a+\|a\|^2 I)$.

Examples

Example 1. (Differentiable function with discontinuous derivative) Consider $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

For $x \neq 0$, we can compute $f'(x)$ by the usual calculus rules.

$$f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$$

At $x = 0$, f' is also well-defined.

$$|f'(0)| = \lim_{h \to 0} \frac{h^2 |\sin \frac{1}{h}|}{|h|} = 0$$

However,

$$\lim_{x \to 0} f'(x) = \lim_{x \to 0} 2x \sin \frac{1}{x} - \cos \frac{1}{x}$$

does not exist. This is because in an arbitrary small neighborhood around $x = 0$, $\cos \frac{1}{x}$ and thus f' will oscillate near 1 and -1. Observe that

$$\cos \frac{1}{x} = \begin{cases} 1 & \text{for } x = \frac{1}{2n\pi} \\ -1 & \text{for } x = \frac{1}{(1+2n)\pi} \end{cases}, n \in \mathbb{Z}$$

Question: How does the foregoing analysis change for $f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$?

Example 2. (Continuous partial derivatives/chain rule) *Find $\varphi'(a)$ for any $a \in \mathbb{R}^m \setminus \{0\}$ if $\varphi(x) := \|x\|, x \in \mathbb{R}^m$.*

Let $\varphi(x) := \|x\| = \sqrt{\sum_{i=1}^m |x_i|^2}$. Then $D_j\varphi(x) = \frac{1}{2}(\sum_{i=1}^m |x_i|^2)^{-\frac{1}{2}}(2|x_j|)\frac{|x_j|}{x_j} = \frac{x_j}{\|x\|}$ is continuous for $x \neq 0$. Therefore, by Theorem 6.2 (Munkres), $\varphi'(a) = \frac{a}{\|a\|}$.

Alternatively, since $\varphi(x) = \sqrt{\|x\|^2}$, the same result can be obtained by the chain rule (which we are yet to prove in $\mathbb{R}^n$) using the result in Question 3.