Quiz Sheet 3

Instructions: Turn in responses to TWO questions of your choice. This quiz will be timed for 15 minutes.

Question 1. Let $\varphi : U \to \mathbb{R}^n$, $U \subseteq R^m$. Suppose U contains a neighborhood of $a \in U$. Let $\varphi_i : U \to \mathbb{R}$ be the i-th component function of φ so that:

$$\varphi(x) = \begin{pmatrix} \varphi_1(x) \\ . \\ . \\ \varphi_n(x) \end{pmatrix}$$

(a) For φ differentiable at $a \in U$, define the Jacobian matrix of φ at a.

(b) Show that $\varphi : U \to \mathbb{R}^n$ is differentiable at $a \iff \varphi_i : U \to \mathbb{R}^n, 1 \leq i \leq n$ is differentiable at a.

(c) Show that if $\varphi : U \to \mathbb{R}^n$ is differentiable at $a \in U$, then $\varphi'(a)$ is the Jacobian matrix.

Question 2. Do all of the following:

(a) Suppose $\varphi : U \to \mathbb{R}^n$. State what it means for φ to be continuously differentiable on U, which we denote by $\varphi \in C^1(U)$.

(b) If $\varphi \in C^1(U)$, what can you say about the map $\varphi' : U \to \mathbb{R}^{n\times m}$

Note: We equip $\mathbb{R}^{n\times m}$ with the metric

$$d(A, B) := \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}(A_{ij} - B_{ij})^2}$$

where A_{ij} and B_{ij} are the elements in the i-th row and j-th column of A and B, respectively.

Question 3. We showed that if $\varphi(x) := \|x\|^2 x,\ x \in \mathbb{R}^m$, then for any $a \in \mathbb{R}^m$,

$$\varphi'(a) = (\|a\|^2 I + 2a \otimes a)$$

In other words, for $h \in \mathbb{R}^m$, we have

$$\varphi'(a)h = (\|a\|^2 I + 2a \otimes a)h = \|a\|^2 h + 2\langle a, h\rangle a$$

Prove that $\varphi(x)$ is of class $C^1(\mathbb{R}^m)$.

Question 4. We showed that for $\varphi(x) := \|x\|,\ x \in \mathbb{R}^m$, for any $a \in \mathbb{R}^m \setminus \{0\}$,

$$\varphi'(a) = \frac{a}{\|a\|}$$

In other words, for $h \in \mathbb{R}^m$, we have

$$\varphi'(a)h = \frac{\langle a, h\rangle}{\|a\|}$$

Determine on which open set $U \subset \mathbb{R}^m$, $\varphi(x)$ is of class $C^1(U)$.

Question 5. Let $U \subseteq \mathbb{R}^n$ be open and $\varphi : U \to \mathbb{R}^n$. Let $b := \varphi(a)$. Suppose that ψ maps a neighborhood of b into $\mathbb{R}^n$ that $\psi(b) = a$ and

$$\psi(\varphi(x)) = x$$

for all x in a neighborhood of a. If φ is differentiable at a, and ψ is differentiable at b, then

$$\psi'(b) = \varphi'(a)^{-1}$$

in $\mathbb{R}^{n\times n}$.