Quiz Sheet 3: Solutions

Question 1. (Proposition 3.2 from Lecture 3) *Let $\varphi : U \to \mathbb{R}^n$, $U \subseteq R^m$. Suppose U contains a neighborhood of $a \in U$. Let $\varphi_i : U \to \mathbb{R}$ be the i-th component function of φ so that:*

$$\varphi(x) = \begin{pmatrix} \varphi_1(x) \\ . \\ . \\ \varphi_n(x) \end{pmatrix}$$

Show that $\varphi : U \to \mathbb{R}^n$ is differentiable at $a \iff \varphi_i : U \to \mathbb{R}^n, 1 \le i \le n$ is differentiable at a. Show that if $\varphi : U \to \mathbb{R}^n$ is differentiable at $a \in U$, then $\varphi'(a)$ is the Jacobian matrix.

See Munkres, p. 46-47.

Question 2. (Question from Lecture 4) *If $\varphi \in C^1(U)$, what can you say about the map $\varphi' : U \to \mathbb{R}^{n\times m}$?*

Note: We equip $\mathbb{R}^{n\times m}$ with the metric

$$d(A, B) := \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}(A_{ij} - B_{ij})^2}$$

where A_{ij} and B_{ij} are the elements in the i-th row and j-th column of A and B, respectively.

By the continuity of each partial derivative $D_j\varphi_i(a)$ for $\frac{\epsilon}{\sqrt{mn}} > 0$, there exists $\delta_{ij} > 0$ such that $\|a - x\| < \delta_{ij}$ implies that $|D_j\varphi_i(a) - D_j\varphi_i(x)| < \frac{\epsilon}{\sqrt{mn}}$. Take $\delta = \min_{1\le i<n, 1\le j\le m} \delta_{ij}$. Then $\|a - x\| < \delta$, implies

$$d(J(a), J(x)) := \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}(D_j\varphi_i(a) - D_j\varphi_i(x))^2} < \epsilon$$

Hence, the Jacobian matrix $J : \mathbb{R}^m \to \mathbb{R}^{n\times m}$ is a continuous at $a \in \mathbb{R}^m$ in the above metric.

Question 3. (The penultimate example from Lecture 4) *We showed that if* $\varphi(x) := ||x||^2 x,\ x \in \mathbb{R}^m$*, then for any* $a \in \mathbb{R}^m$,

$$\varphi'(a) = (||a||^2 I + 2a \otimes a)$$

In other words, for $h \in \mathbb{R}^m$*, we have*

$$\varphi'(a)h = (||a||^2 I + 2a \otimes a)h = ||a||^2 h + 2\langle a, h\rangle a$$

Prove that $\varphi(x)$ *is of class* $C^1(\mathbb{R}^m)$.

By Question 1, $\varphi'(a)$ is the Jacobian matrix, which implies that the i-th row vector is given by

$$\varphi_i'(a) = (D_1\varphi_i(a), ..., D_m\varphi_i(a)) = ||a||^2 e_i^T + 2a_i a$$

Therefore,

$$D_j\varphi_i'(x) = ||x||^2 \delta_{ij} + 2x_i x_j$$

is continuous at a. The demonstration of continuity of a function given by $||x||^2$ is straightforward. x_i depends continuously on x as well: $||a - x|| < \epsilon \Rightarrow ||a_i - x_i|| < \epsilon$. Lastly sums and products of continuous real-valued functions are continuous (e.g., Munkres, Theorem 3.6).

Question 4. (The last example from Lecture 4) *We showed that for* $\varphi(x) := ||x||,\ x \in \mathbb{R}^m$*, for any* $a \in \mathbb{R}^m \setminus \{0\}$,

$$\varphi'(a) = \frac{a}{||a||}$$

In other words, for $h \in \mathbb{R}^m$*, we have*

$$\varphi'(a)h = \frac{\langle a, h\rangle}{||a||}$$

Determine on which open set $U \subset \mathbb{R}^m$*,* $\varphi(x)$ *is of class* $C^1(U)$.

Similarly to the preceding problem,

$$\varphi'(a) = (D_1\varphi(a), ..., D_m\varphi(a)) = \frac{a}{||a||}$$

Therefore,

$$D_j\varphi'(x) = \frac{x_j}{||x||}$$

is continuous at $a \in \mathbb{R} \setminus \{0\}$ (a quotient of continuous real-valued functions).

We demonstrate the partial derivatives are not well-defined at zero.

$$D_j\varphi'(0) = \lim_{h\to 0} \frac{\sqrt{h^2}}{h} = \begin{cases} 1 \text{ if } h > 0 \\ -1 \text{ if } h < 0 \end{cases}$$

Question 5. (The last exercise from Lecture 5) *Let $U \subseteq \mathbb{R}^n$ be open and $\varphi : U \to \mathbb{R}^n$. Let $b := \varphi(a)$. Suppose that ψ maps a neighborhood of b into $\mathbb{R}^n$ that $\psi(b) = a$ and*

$$\psi(\varphi(x)) = x$$

for all x in a neighborhood of a. If φ is differentiable at a, and ψ is differentiable at b, then

$$\psi'(b) = \varphi'(a)^{-1}$$

in $\mathbb{R}^{n\times n}$.

See Munkres, p 60.