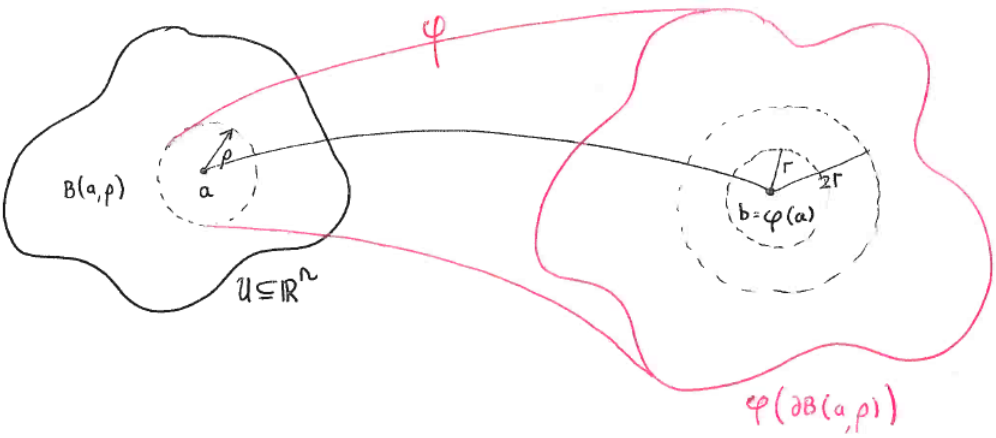
Quiz Sheet 5

Instructions: Turn in responses to TWO questions of your choice. This quiz will be timed for 15 minutes.

Question 1. Referring to the figure below, for $y \in B(b, r)$, we define the map $\Phi_y : U \to \mathbb{R}$ by

$$\Phi_y(x) := \|\varphi(x) - y\|^2, \ x \in U$$

where $U \subseteq \mathbb{R}^n$ is open, $\varphi : U \to \mathbb{R}^n$ is of class $C^1(U)$ and one-to-one on U, $\varphi'(x)$ is nonsingular for all $x \in U$, and $a \notin \partial B(a, \rho)$. Explain why $\Phi_y(a) := \|\varphi(a) - y\|^2 = \|b - y\| < r^2$ (and not $\leq r^2$).



Question 2. Refering to the hypothesis of Question 1, do all of the following.

(a) Define a function ψ such that $\Phi_y = \psi \circ \phi$, and specify the domain and the range of ψ.

(b) Using the chain rule, confirm that Φ_y is of class $C^1(U)$.

(c) Deduce that $D\Phi_y(x_{min}, e_i) = 0$ for $i = 1, ..., n$, where $x_{min} \in B(a, \rho)$ and $D\Phi_y(\cdot, e_i)$ is a directional derivative in the direction of the i-th canonical basis vector of $\mathbb{R}^n$, if and only if

$$\sum_{k=1}^{n} 2(\varphi_k(x_{min}) - y_k) D_i \varphi_k(x_{min}) = 0$$

for $i = 1, ..., n$.

Question 3. Do all of the following:

(a) State what it means for a nonempty set V, $+ : V^2 \to V$ and $\cdot : \mathbb{R} \times V \to V$ to be a *real vector space*.

(b) Let

$$V := \{p : \mathbb{R} \to \mathbb{R} \mid p(x) = \sum_{k=0}^{n} a_k x^k,\ x \in \mathbb{R},\ \text{for some } a_k \in \mathbb{R}\}$$

Show that V is real vector space and show that the dimension of V is $n + 1 \in \mathbb{N}$

Hint: Define $+ : V^2 \to V$ *and* $\cdot : \mathbb{R} \times V \to V$ *first.*

Question 4. Let $V := \mathbb{R}^{n \times n}$. Show that V is a real vector space. Show that the dimension of V is $n^2 \in \mathbb{N}$.

Question 5. Let $V := C(\mathbb{R}, \mathbb{R})$ be the set of all continuous functions on $\mathbb{R}$ with range in $\mathbb{R}$. Show that V is a real vector space. What can you say about the dimension of V?

Question 6. Do all of the following:

(a) If V is a real vector space, state what it means for a subset $W \subseteq V$ to be a *subspace* of V.

(b) Let $W := C^1(\mathbb{R}, \mathbb{R})$. Show that W is a subspace of $V := C(\mathbb{R}, \mathbb{R})$. ($C(\mathbb{R}, \mathbb{R})$ is defined in Question 5.)