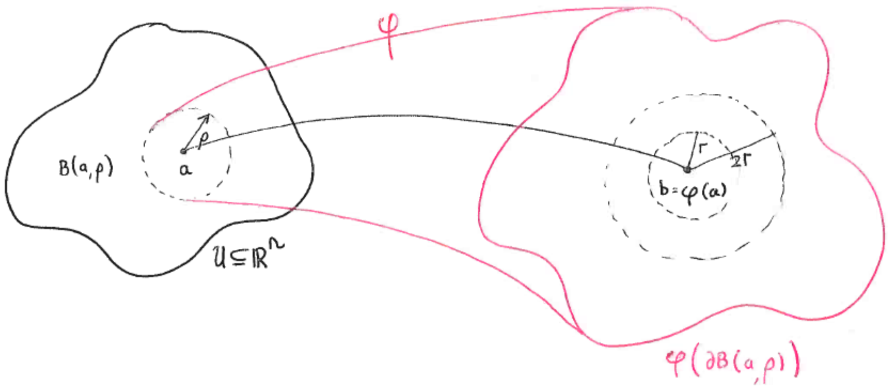
Quiz Sheet 5: Solutions

Question 1. (from Lecture 7, p. 7) *Referring to the figure below, for $y \in B(b,r)$, we define the map $\Phi_y : U \to \mathbb{R}$ by*

$$\Phi_y(x) := \|\varphi(x) - y\|^2, \; x \in U$$

where $U \subseteq \mathbb{R}^n$ is open, $\varphi : U \to \mathbb{R}^n$ is of class $C^1(U)$ and one-to-one on U, $\varphi'(x)$ is nonsingular for all $x \in U$, and $a \notin \partial B(a,\rho)$. Explain why $\Phi_y(a) := \|\varphi(a) - y\|^2 = \|b - y\|^2 < r^2$ (and not $\leq r^2$).



If $\|b - y\| = r$, this implies that $y \notin B(b,r)$.

Question 2. (from Lecture 7, pp. 8-9) *Referring to the hypothesis of Question 1, do all of the following.*

(a) *Define a function ψ such that $\Phi_y = \psi \circ \varphi$, and specify the domain and the range of ψ.*

$$\psi : \mathbb{R}^n \to \mathbb{R}$$

defined by $\psi(s) = \|s - y\|^2$

(b) *Using the chain rule, confirm that Φ_y is of class $C^1(U)$.*

We have

$$D\Phi_y = D\psi(\varphi) \cdot D\varphi = 2\varphi \cdot D\varphi$$

where the last equality follows from the last Exercise from Lecture 2, part (a). $D\Phi_y$ is continuous since φ is of class $C^1(U)$.

(c) *Deduce that* $D\Phi_y(x_{min}, e_i) = 0$ *for* $i = 1, ..., n$, *where* $x_{min} \in B(a, \rho)$ *and* $D\Phi_y(\cdot, e_i)$ *is a directional derivative in the direction of the i-th canonical basis vector of* $\mathbb{R}^n$, *if and only if*

$$\sum_{k=1}^{n} 2(\varphi_k(x_{min}) - y_k) D_i \varphi_k(x_{min}) = 0$$

for $i = 1, ..., n$.

Since $D\Phi(x_{min})$ and $D\varphi(x_{min})$ are Jacobian matrices, we have $D\Phi(x_{min}, e_i) = D_i\Phi(x_{\text{min}})$ and $D\varphi(x_{min}, e_i) = D_i\varphi(x_{\text{min}})$, which are the i-th column vectors of the corresponding Jacobians. Therefore,

$$\begin{aligned} D\Phi_y(x_{min}, e_i) =& D_i\Phi_y(x_{\text{min}}) = 2(\varphi(x_{\text{min}}) - y) \cdot D_i\varphi(x_{\text{min}}) \\ =& \sum_{k=1}^{n} 2(\varphi_k(x_{min}) - y_k) D_i\varphi_k(x_{min}) = 0 \end{aligned}$$

Question 3. (from Lecture 8, Example 8.3) *Let*

$$V := \{p : \mathbb{R} \to \mathbb{R} \mid p(x) = \sum_{k=0}^{n} a_k x^k,\ x \in \mathbb{R},\ \text{for some } a_k \in \mathbb{R}\}$$

Show that V is real vector space and show that the dimension of V is $n+1 \in \mathbb{N}$.

See Solution to Problem 1 from Homework 5.

Question 4. (from Lecture 8, Example 8.4) *Let* $V := \mathbb{R}^{n \times n}$. *Show that V is a real vector space. Show that the dimension of V is* $n^2 \in \mathbb{N}$.

(Sketch of proof) The closure under addition and scalar multiplication and the other vector space properties follow from the canonical addition and scalar multiplication $+ : V^2 \to V$ by $A_1 + A_2$ where the ij-th entry of the sum is given by $(A_1 + A_2)_{ij} = a_{1ij} + a_{2ij}$ and a_{1ij} and a_{2ij} are the ij-th entries of A_1 and A_2 respectively, and $\cdot : \mathbb{R} \times V \to V$ by λA where the ij-th entry of the is given by $(\lambda A)_{ij} = \lambda a_{ij}$ where a_{ij} are the ij-th entries of A.

It is also straightforward to show that the set of $n \times n$ matrices E_{ij} that have all zero entries except for the ij-th entry, where $1 \leq i, j \leq n$ are linearly independent and span V. There are n^2 such matrices.

Question 5. (from Lecture 8, Example 8.5) *Let $V := C(\mathbb{R}, \mathbb{R})$ be the set of all continuous functions on $\mathbb{R}$ with range in $\mathbb{R}$. Show that V is a real vector space. What can you say about the dimension of V?*

The proof is the same as the proof that the set of all maps $f : \mathbb{R}^m \to \mathbb{R}^n$ of class $C^1(\mathbb{R}^m)$ is a real vector space in Problem 2 of Homework 5 (taking $m = n = 1$) except that closure follows from the basic result that linear combinations of continuous functions are continuous (Munkres, Theorem 3.6).

Question 6. (from Lecture 8, Example 8.6)*Let $W := C^1(\mathbb{R}, \mathbb{R})$. Show that W is a subspace of $V := C(\mathbb{R}, \mathbb{R})$. ($C(\mathbb{R}, \mathbb{R})$ is defined in Question 5.)*

It suffices to prove that W is closed under addition and scalar multiplication, which is the same as the proof that the set of all maps $f : \mathbb{R}^m \to \mathbb{R}^n$ of class $C^1(\mathbb{R}^m)$ is closed under addition and scalar multiplication in Problem 2 of Homework 5 .