Quiz Sheet 6: Solutions

Question 1. *(Munkres, Ex. 6, p. 226)*

(b) *Let f and g be the following tensors on $\mathbb{R}^4$:*

$$
\begin{aligned}
f(x, y, z) &= 2x_1y_2z_2 - x_2y_3z_1 \\
g &= \phi_{2,1} - 5\phi_{3,1}
\end{aligned}
$$

Express $f \otimes g$ as a linear combination of elementary 5-tensors.

By Lemma 26.2 (Munkres), we have

$$f = 2\phi_{1,2,2} - \phi_{2,3,1}$$

Therefore,

$$
\begin{aligned}
f \otimes g &= (2\phi_{1,2,2} - \phi_{2,3,1}) \otimes (\phi_{2,1} - 5\phi_{3,1}) \\
&= 2\phi_{1,2,2,2,1} - 10\phi_{1,2,2,3,1} - \phi_{2,3,1,2,1} + 5\phi_{2,3,1,3,1}
\end{aligned}
$$

(c) *Express $(f \otimes g)(x, y, z, u, v)$ as a function.*

By Lemma 26.2 (Munkres), for $v, w \in \mathbb{R}^4$, we have

$$g(u, v) = u_2v_1 - 5u_3v_1$$

Therefore,

$$(f \otimes g)(x, y, z, u, v) = (2x_1y_2z_2 - x_2y_3z_1)(u_2v_1 - 5u_3v_1)$$

Question 2. *(Munkres, Ex. 1, p. 236) Which of the following are alternating tensors in $\mathbb{R}^4$?*

$$f(x, y) = x_1y_2 - x_2y_1 + x_1y_1 \tag{1}$$

is not an alternating tensor since

$$
\begin{aligned}
f(x, y) + T_\sigma f(x, y) &= f(x, y) + f(y, x) \\
&= (x_1y_2 - x_2y_1 + x_1y_1) + (y_1x_2 - y_2x_1 + y_1x_1) \\
&= 2x_1y_1 \neq 0
\end{aligned}
$$

$$g(x, y) = x_1 y_3 - x_3 y_2 \tag{2}$$

is not an alternating tensor since

$$g(x, y) + T_\sigma g(x, y) = g(x, y) + g(y, x) = x_1 y_3 - x_3 y_2 + y_1 x_3 - y_3 x_2 \neq 0$$

$$h(x, y) = (x_1)^3 (y_2)^3 - (x_2)^3 (y_1)^3$$

is not an alternating tensor since it is not multilinear, e.g.,

$$h(2x, y) = (2x_1)^3 (y_2)^3 - (2x_2)^3 (y_1)^3 = 8h(x, y) \neq 2h(x, y)$$

Question 3. *Show that* $f \wedge g \wedge f = 0$ *where f and g are the following alternating tensors in* $\mathbb{R}^2$*:*

$$f(x, y) = x_1 y_2 - x_2 y_1$$
$$g(x, y) = -x_1 y_2 + x_2 y_1$$

$f \wedge g \wedge f \in A^6(\mathbb{R}^2)$. Therefore, by Problem 7 from Homework Sheet 6, we have $f \wedge g \wedge f = 0$.