Quiz 8: Solutions

Question 1. *Find the exterior derivative* $d\omega(x_1 y_1)$. *Evaluate* $d\omega(x)((x,v))$ *for* $x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ *and* $v = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

By definition of the exterior derivative of a 0-form, $d\omega$ is given by

$$(d\omega(x))((x,v)) := \omega'(x)v = (x_2, x_1) \cdot v$$

By Proposition 2.2 from the April 18th lecture (Thm 30.3 Munkres)

$$d\omega = \frac{\partial \omega}{\partial x_1} dx_1 + \frac{\partial \omega}{\partial x_2} dx_2$$

where $dx_i = \Phi_1$ are the elementary 1-forms in $\Omega^1(\mathbb{R}^n)$. Therefore,

$$(d\omega(x))((x,v)) := \omega'(x)v = (2,1) \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix} = 10$$

Question 2. *Find the exterior derivative* $d\omega$ *of* ω *given by* $\omega(x) = x_1 x_2 dx_1$. *Evaluate* $d\omega(x)((x,v_1),(x,v_2))$ for $x = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $v_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

By definition of a derivative of a k-form for $k \geq 1$, we have

$$d\omega = \sum_{I \text{ ascending}} df_I \wedge dx_I$$

where $f_I : U \subseteq \mathbb{R}_n$ are given by the following representation for all k forms

$$\omega = \sum_{I \text{ ascending}} f_I dx_I$$

as established in a previous lecture (see also Munkres p. 249).

Here we have $I = i$ where $i \in \{1,2\}$, $f_1 = x_1 x_2$ and $f_2 = 0$. By definition of the exterior derivative of 0-form and Proposition 2.2, we have $df_1 = (x_2 dx_1 + x_1 dx_2)$, and $df_2 = 0$. Consequently,

$$d\omega = (x_2 dx_1 + x_1 dx_2) \wedge dx_1 = x_1 dx_2 \wedge dx_1 = -x_1 dx_1 \wedge dx_2$$

where $dx_1 \wedge dx_2 = 0$ follows from the anticommutativity of the wedge product.

This implies that

$$d\omega(x)((x;v_1),(x;v_2)) = 0$$

Question 3. (Munkres, Ex 2, p. 260) *Consider the forms*

$$\omega = xydx + 3dy - yzdz$$
$$\eta = xdx - yz^2dy + 2xdz$$

$\in \mathbb{R}^3$. *Verify by direct computation that* $d(d\omega) = 0$ *and* $d(\omega\wedge\eta) = (d\omega)\wedge\eta - \omega\wedge d\eta$.

By definition of the exterior derivative of 0-form and Proposition 2.2, we have

$$\begin{aligned} d\omega &= (ydx + xdy) \wedge dx - (zdy + ydz) \wedge dz \\ &= xdy \wedge dx - zdy \wedge dz \\ &= -xdx \wedge dy - zdy \wedge dz \end{aligned}$$

and,

$$\begin{aligned} d(d\omega) &= d(-xdx \wedge dy - zdy \wedge dz) \\ &= d(-xdx \wedge dy - zdy \wedge dz) \\ &= -dx \wedge dx \wedge dy - dz \wedge dy \wedge dz \\ &= 0 \end{aligned}$$

The LHS is given by

$$\begin{aligned} \omega \wedge \eta &= (xydx + 3dy - yzdz) \wedge (xdx - yz^2dy + 2xdz) \\ &= -xy^2z^2dx \wedge dy + 2x^2ydx \wedge dz + 3xdy \wedge dx + 6xdy \wedge dz \\ &\quad - xyzdz \wedge dx + y^2z^3dz \wedge dy \\ &= -(xy^2z^2 + 3x)dx \wedge dy + (2x^2y + xyz)dx \wedge dz + (6x - y^2z^3)dy \wedge dz \end{aligned}$$

and, therefore,

$$\begin{aligned} d(\omega \wedge \eta) &= -((y^2z^2 + 3)dx + 2xyz^2dy + 2xy^2zdz) \wedge dx \wedge dy \\ &\quad + ((4xy + yz)dx + (2x^2 + xz)dy + xydz) \wedge dx \wedge dz \\ &\quad + (6dx - 2yz^3dy - 3y^2z^2dz] \wedge dy \wedge dz \\ &= -2xy^2zdz \wedge dx \wedge dy + (2x^2 + xz)dy \wedge dx \wedge dz + 6dx \wedge dy \wedge dz \\ &= (-2x^2 - 2xy^2z - xz + 6)dx \wedge dy \wedge dz \end{aligned}$$

The RHS is given by

$$
\begin{aligned}
(d\omega)\wedge\eta-\omega\wedge d\eta =& d(xydx+3dy-yzdz)\wedge(xdx-yz^2dy+2xdz)\\
&-(xydx+3dy-yzdz)\wedge d(xdx-yz^2dy+2xdz)\\
=&(d(xy)\wedge dx-d(yz)dz)\wedge(xdx-yz^2dy+2xdz)\\
&-(xydx+3dy-yzdz)\wedge(d(x)dx-d(yz^2)dy+d(2x)dz)\\
=&((ydx+xdy)\wedge dx-(zdy+ydz)\wedge dz)\wedge(xdx-yz^2dy+2xdz)\\
&-(xydx+3dy-yzdz)\wedge(dx\wedge dx-(z^2dy+2yzdz)\wedge dy+2dx\wedge dz)\\
=&(xdy\wedge dx-zdy\wedge dz)\wedge(xdx-yz^2dy+2xdz)\\
&-(xydx+3dy-yzdz)\wedge(-2yzdz\wedge dy+2dx\wedge dz)\\
=&2x^2dy\wedge dx\wedge dz-xzdy\wedge dz\wedge dx\\
&-(-2xy^2zdx\wedge dz\wedge dy+6dy\wedge dx\wedge dz)\\
=&(-2x^2-2xy^2z-xz+6)dx\wedge dy\wedge dz
\end{aligned}
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