Quiz Sheet 9

Instructions: Turn in responses to TWO questions of your choice. This quiz will be timed for 15 minutes.

Question 1. Let A be open in $\mathbb{R}^3$. Let $f : A \to \mathbb{R}$ be a scalar field (a/k/a the differential 0-form). We define a corresponding vector field in A, called the *gradient* of f, by

$$(\text{grad } f)(x) = \nabla f(x) = (x; \frac{\partial f(x)}{\partial x_1}e_1 + \frac{\partial f(x)}{\partial x_2}e_2 + \frac{\partial f(x)}{\partial x_3}e_3)$$

Let α_1 be a map from the set of all C^∞ vector fields on A (a/k/a $\Omega^0(A)$) to $\Omega^1(A)$ given by

$$\alpha_1 F = f_1 dx_1 + f_2 dx_2 + f_2 dx_2$$

where the vector field F is given by

$$F(x) = (x; f_1(x)e_1 + f_2(x)e_2 + f_3(x)e_3)$$

Show that $d = \alpha_1 \circ$ grad.

Hint: To evaluate the LHS, apply the definition of the exterior derivative of a 0*-form and to evaluate the RHS apply the definitions given above.*

Question 2. Let A be open in $\mathbb{R}^3$, and F be a vector field in A given by $F(x) = (x; f_1(x)e_1 + f_2(x)e_2 + f_3(x)e_3)$ for $x \in A$. We define another vector field in A, called the *curl* of f, by

$$(\text{curl } F)(x) = \nabla \times F(x) = (x; (\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3})e_1 + (\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1})e_2 + (\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2})e_3)$$
$$= \det \begin{pmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ f_1 & f_2 & f_3 \end{pmatrix}.$$

where each partial derivative is evaluated at $x \in A$. Let β_2 be a map from the set of all C^∞ vector fields on A to $\Omega^2(A)$ given by

$$\beta_2 F = f_1 dx_2 \wedge dx_3 - f_2 dx_1 \wedge dx_3 + f_3 dx_1 \wedge dx_2$$

Show that $d \circ \alpha_1 = \beta_2 \circ$ curl where α_1 is as defined in Question 1.

Hint: To evaluate the LHS, apply the definition of the exterior derivative of a 1*-form to* $\alpha_1 F$*. To evaluate the RHS expand* $\beta_2 \circ$ *curl* F *using the definitions given above.*

Question 3. Let A be open in $\mathbb{R}^3$. Let $G(x) = (x; g(x))$ be a vector field in A, where $g : A \to \mathbb{R}^3$ is given by

$$g(x) = g_1(x)e_1 + g_2(x)e_2 + g_3(x)e_3$$

we define a corresponding scalar field in A, called the *divergence* of f, by

$$(\text{div } G)(x) = \nabla \cdot G(x) = \frac{\partial g_1(x)}{\partial x_1} + \frac{\partial g_2(x)}{\partial x_2} + \frac{\partial g_3(x)}{\partial x_3}$$

Let β_3 be a map from the set of all C^∞ scalar fields on A to $\Omega^3(A)$ given by

$$\beta_3 h = h dx_1 \wedge dx_2 \wedge dx_3$$

where h is a scalar field. Show that $d \circ \beta_2 = \beta_3 \circ \text{ div}$.