MATH 3345 REVIEW PROBLEMS FOR MIDTERM 1

We will discuss these problems on Monday, October 1 in class. You should work on as many of these problems as possible before our review session.

0. Read the last subsection in §2.4 ("General Guidelines for Constructing Proofs").

1. Suppose P is false and that the statement $(R \Rightarrow S) \Leftrightarrow (P \wedge Q)$ is true. Find the truth values of R and S without writing out a truth table.

2. Replace the following statements by an equivalent statement that uses fewer connectives:
(a) $\neg(P \Rightarrow \neg Q)$
(b) $P \Rightarrow \neg P$

3. Determine whether the following pairs of statements are logically equivalent. Explain your answers.
(a) $P \wedge Q$ and $\neg(\neg P \vee \neg Q)$
(b) $P \Rightarrow Q$ and $(P \wedge \neg Q) \Rightarrow (Q \wedge \neg Q)$

4. Determine whether the following statements are true or false, assuming all variables have the set of real numbers $\mathbb{R}$ as their domain. Justify your answers.
(a) $\forall x\, \exists y\, (x^2 = y)$
(b) $\forall y\, \exists x\, (x^2 = y)$
(c) $\exists x\, \forall y\, (x + 5 = y)$
(d) $\forall x\, \forall y\, \exists z\, (x^2 + y^2 = z^2)$
(e) $\exists x\, [\forall y\, (yx^2 = y) \wedge \neg \forall y\, (yx = y)]$

5. What is the converse of $(P \Rightarrow (Q \Rightarrow R)) \Rightarrow ((P \wedge Q) \Rightarrow R)$? What is the contrapositive?

6. Give a precise definition of a prime number, and prove that there are infinitely many primes.

7. Prove that $n^2 + 3n + 4$ is even for all integers n. (Do not use induction.)

8. Give a precise definition of a rational number, and prove that $\sqrt[3]{2}$ is irrational.

9. Prove or disprove the following statements.
(a) If $a \equiv 0 \pmod{m}$ and $b \equiv 1 \pmod{n}$, then $ab \equiv a \pmod{mn}$.
(b) If $a \equiv 3 \pmod{11}$ and $b \equiv 7 \pmod{11}$, then $11|(3a + 5b)$.
(c) There exist integers $a \equiv 3 \pmod{11}$ and $b \equiv 7 \pmod{11}$ such that $11|(5a + 2b)$.
(d) If $a \equiv b \pmod{5}$ and $c \equiv d \pmod{7}$, then $ac \equiv bd \pmod{35}$.

10. Prove that a positive integer is divisible by 9 if and only if the sum of its digits (when expressed in base 10) is divisible by 9.

11. Find an integer k with the following property: if you add k times the last digit of a positive integer to the number represented by the remaining digits, then the result is divisible by 23 if and only if the original number is divisible by 23. (Prove your answer.)

12. Using the fact that π is irrational, show that $3 + \sqrt{\pi}$ or $3 - \sqrt{\pi}$ is irrational.