FLOW-CUT DUALITIES FOR SHEAVES ON GRAPHS

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ABSTRACT. This paper generalizes the Max-Flow Min-Cut (MFMC) theorem from the setting of numerical capacities to cellular sheaves of semimodules on directed graphs. Motivating examples of semimodules include probability distributions, multicommodity capacity constraints, and logical propositions. Directed algebraic topology provides the tools necessary for capturing the salient information in such a general setting. First homology classes generalize flows, an orientation sheaf characterizes generalized cuts, first relative homology measures duality gaps, zeroth homology classes generalize both flow-values and cut-values, and inverse limits generalize infima. Under this dictionary, MFMC is just a special case of a Poincaré Duality for directed topology. A Universal Coefficients Theorem for directed homology generalizes existing criteria for monoid-valued flows to decompose into sums of generalized loops. First homology coincides with a standard generalization of Abelian homology for non-Abelian categories under an assumption of stalkwise flatness, stalkwise module structure, or certain degree bounds on the vertices.

1. Introduction

Sheaves encode local constraints. Abelian sheaf cohomology, by definition, classifies those global properties of a sheaf of modules invariant under equivalent local representations of the same data. Abelian sheaf cohomology has seen recent applications in the inference of global properties of complex systems with known local structure. Examples include bit-rates across coding networks [6], minimum sampling rates for noisy signals [9], and invariant states and race conditions on asynchronous microprocessors [9]. However, the sectionwise invertibility of Abelian sheaves ignores the irreversibility of states in dynamical systems. For example, the (co)homology of a cellular module-valued sheaf on an oriented simplicial complex is invariant under a change in simplicial orientations; properties of systems sensitive to the causal structure of their state spaces are undetectable by classical sheaf (co)homology.

Nonetheless, flow-cut dualities resemble topological dualities. For one example, to each minimal cut corresponds a maximal flow such that the corresponding induced cohomology and homology classes on an ambient compact surface are Poincaré dual [3]. For another example, a version of Poincaré Duality for sheaves of vector spaces implicitly underlies an analysis of distributed linear coding [6]. For another example, the proof of the classical Max-Flow Min-Cut theorem (MFMC) on directed graphs satisfying the natural graph-theoretic versions of compactness, orientability, and smoothness is trivial. For still another example, a recent general proof of classical MFMC follows from the Riemann-Roch Theorem [1]. Flows resemble homology classes, cuts resemble cohomology classes, local capacity constraints resemble a sheaf, and flow-cut dualities evoke the Poincaré duality

$$H_1(X;\mathcal{F}) \cong H^0(X;\mathcal{O}\otimes\mathcal{F}). \tag{1}$$

between sheaf homology $H_1(X;\mathcal{F})$ and sheaf cohomology $H^0(X;\mathcal{O}_S\otimes_S\mathcal{F})$ up to local orientations $\mathcal{O}$ for $\mathcal{F}$ a sheaf of modules over a topological graph X [Theorem ...].

This note formalizes that resemblance by generalizing the constructions in (1). Local constraints on networks generalize to *cellular sheaves $\mathcal{F}$ of (partial) semimodules* on directed graphs X. Homological constructions for such sheaves generalize familiar constructions on networks. First directed homology H_1 generalizes flows [Theorem 6.12], an inverse limit of H^0 over cut-sets lifts taking infima of cut-values, zeroth directed homology H_0 (stalks up to parallel transport) generalizes flow-values [Propositions ...] and cut-values [Propositions ...], boundary maps $\partial_-,\partial_+ : H_1((X,C);\mathcal{F})\to H_0(C;\mathcal{F})$ reduce flows and cuts to flow-values and cut-values [Propositions ...], and parallel transport makes it possible to compare values of a flow at an edge e with values at an e-cut. A limited Universal Coefficients Theorem for directed homology [Proposition 6.10] identifies criteria for generalized flows to decompose into generalized loops. Additional lattice structure makes it possible to pose dual optimization problems for a sheaf. The main result of this note is a generalization of classical MFMC for sheaf-valued flows over a directed graph.

Theorem 7.9. *There exists an isomorphism*

$$\underbrace{[e:X]_{\mathcal{F}}}_{max\ \mathcal{F}\text{-}flow\ value}\cong \inf{}_C \underbrace{[C:C]_{\mathcal{F}}}_{\mathcal{F}\text{-}values\ of\ e\text{-}cuts\ C}\quad , \tag{2}$$

where C denotes an e-cut, for each hard lattice-ordered S-sheaf $\mathcal{F}$ on X and edge e in X.

A consequence is a decomposition of the feasible set of flow-values as an intersection of feasible local flow-values over cut-sets.

Corollary 7.11. *There exists an isomorphism*

$$[e:X]_{spec\,\mathcal{F}}\cong\bigcap_C [C:C]_{spec\,\mathcal{F}}, \tag{3}$$

where C ranges over all e-cuts of X, for each hard but flat S-sheaf and $e\in E_X$.

Another consequence is an algebraic generalization of MFMC for totally ordered $\mathbb{N}$-semimodules [5], and hence in particular classical MFMC [4].

Corollary 7.12. *For a finite M-weighted digraph $(X;\omega)$ with edge e_0,*

$$\sup{}_\phi \phi(e_0)=\inf{}_C\sum_{e\in C}\omega_e,$$

where ϕ denotes a M-valued flow ϕ on $(X;\omega)$ and C denotes an e_0-cut.

2. Outline

Modules over commutative monoid objects S in monoidal categories are ubiquitous in the literature. Such *S-semimodules* include classical semimodules (over rings S), more general classical semimodules (over semirings S), and even more general *partial semimodules* (over *partial semirings S*). *Partial semimodules*, sets equipped with partially defined additions and scalar multiplications, will later encode capacity constraints on the individual edges of a network. Section §4 introduces some of the theory of semimodules, including a description of limits for partial semimodules

[Proposition ...] and a characterization of partial semimodules as ideal complements in classical semimodules [...].

An S-*sheaf*, a cellular sheaf of S-semimodules over a directed graph, generalizes edge weights. For example, an *orientation sheaf* $\mathcal{O}_S$ [Definition 5.2] over a general commutative object S in a monoidal category measures singularities of the directed graph [Lemma 5.8 and Figure 5] in a sense determined by the choice of commutative monoid object S. The stalkwise freeness of $\mathcal{O}_{\mathbb{N}}$ detects bounds on local in-degrees and out-degrees [Lemma 5.5 and Figure 6.18]. The stalks of $\mathcal{O}_R$ are just the ordinary local homology modules [Proposition 5.4] over a ring R and hence are free and invariant under a change in edge directions. Section §5 introduces the theory of S-sheaves.

(Co)homology theories for sheaves of modules generalize for S-sheaves [7]. The *global sections functor* sends a sheaf $\mathcal{F}$ to the limit $\lim_c \mathcal{F}(c)$ over the vertices and edges of the base graph. Like the compactly supported global sections functor H^0_c, different subfunctors H^0 of the global sections functor yield different variants of directed (co)homology for S-sheaves. Categorical limits of partial semimodules are not just ordinary limits of underlying sets with extra structure. This note introduces an *everywhere defined global sections* functor H^0_∞, for sheaves of partial semimodules, to circumvent such pathologies. Section §6.1 introduces the theory, and in particular investigates the interaction of H^0_∞ with sheaf tensor products [Lemma 3.21]

Homology for sheaves classifies global twisted cycles up to global homotopies between such cycles. Zeroth directed homology amounts to products of stalks modulo parallel transport. On a graph, the tensor sheaf $\mathcal{F} \otimes_S \mathcal{O}_S$ essentially describes the local twisted 1-cycles and the trivial sheaf essentially describes the homotopy relations between local 1-cycles. Thus first directed homology on graphs [Definition 6.8] is effectively defined by Poincaré Duality (1). Directed homology, unlike zeroth directed (co)homology, depends upon the choice of ground semiring. First directed homology with constant semiring coefficients S classifies directed loops for $S = \mathbb{N}$ [Theorem 6.12] and undirected loops for $S = \mathbb{Z}$ [Theorem 6.12]. Natural maps

$$\partial_-, \partial_+ : H_1((X,C);\mathcal{F}) \to H_0(C;\mathcal{F})$$

collectively generalize the connecting homomorphism for ordinary homology [Proposition ...]. Unlike in the classical case of a ground ring, ∂_-, ∂_+ do not fit into a natural generalization of an exact sequence for directed homology [Example ...] Sections §6.2 and §6.3 introduce H_0, H_1, and section §6.4 introduces the connecting homomorphisms.

The following limited version of the Universal Coefficients Theorem generalizes for the directed setting.

Proposition 6.10 (Universal Coefficients)**.** *There exists an S-map*

$$H_1(X;\mathcal{F}) \otimes_S M \cong H_1(X;\mathcal{F} \otimes_S k_M)$$

natural in S-sheaves $\mathcal{F}$ and S-semimodules M and an isomorphism for M flat, where $\mathcal{F} \otimes_S k_M$ is regarded as an S-semimodule.

Chain-theoretic constructions of homology generalize for the directed setting. Under certain local algebraic or local geometric criteria, first directed homology coincides with a degree 1 homology theory for higher categorical structures [8] and thus admits an intuitive interpretation as *sheaf-valued flows*, sheaf-valued chains satisfying a conservation law expressed in terms of an equalizer diagram.

Theorem 6.12. *For a locally finite digraph X, there exists an equalizer diagram*

$$H_1(X;\mathcal{F}) \dashrightarrow \prod_{e\in E_X} \mathcal{F}(\partial_- e) \times_{\mathcal{F}(e)} \mathcal{F}(\partial_+ e) \underset{\pi_+}{\overset{\pi_-}{\rightrightarrows}} \prod_{v\in V_X} \mathcal{F}(v),$$

with π_-, π_+ induced by projections onto first and second factors, for an S-sheaf $\mathcal{F}$ on X such that $\mathcal{F}$ is flat, S is a ring, or each vertex in X has in-degree or out-degree 1.

Classical MFMC generalizes to a sheaf-theoretic setting. Section §7 details the generalization. Sheaves of (partial) semimodules naturally encode numeric capacity constraints [Example 7.1] on transportation networks, multicommodity constraints [Example 7.2] on supply chains, and even logical constraints [Example 7.3]. Classical flows naturally generalize to sheaf-valued flows [Proposition 7.4], classified by directed sheaf homology under the assumption of flatness. Classical directed cuts naturally admit a characterization in terms of the orientation sheaf [Proposition 7.8]. The generalized connecting homomorphisms ∂_-, ∂_+ send flows to their values [Proposition ...] Additional order-structure on the coefficient sheaf makes it possible to define maximum flow-values and minimum cut-values. Theorem 7.9 decomposes the suprema of e-values $[e : X]_{\mathcal{F}}$ of $\mathcal{F}$-flows as an infimum over cut-values $[C : C]_{\mathcal{F}}$ of e-cuts C.

The proof of Theorem 7.9 requires two steps. *Weak duality*, an upper bound on the maximum flow-value given as the infima of cut-values, directly follows from tensorial properties of *locally hard* sheaves and a cohomological characterization of cuts [Proposition 7.8]. The proof that the maximum flow-value bounds the minimum cut-value follows from local-to-global properties, reminiscent of descent in simplicial sheaves, of *lattice-ordered* sheaves. In particular, there exists an operator on the Cech 0-cochains of such sheaves - defined in terms of infima operations, that increasingly approximates such Cech 0-cochains by Cech 0-cocycles, global sections. The Tarski Fixed Point Theorem for complete lattices implies the existence of a desired maximal flow.

Throughout, the note adopts the following general conventions. The cardinality of a set X is written $\#X$. This note ocassionally abuses notation and conflates an element x in a set with its singleton set $\{x\}$ and in particular sometimes lets $X - x$ denote the set $X - \{x\}$. Additionally, the following section §3 fixes some notation and terminology for directed graphs.

3. Digraphs

This note takes *digraph* to mean a reflexive directed graph, a directed graph allowing for self-loops at the vertices. For a digraph X, V_X denotes its vertex set, E_X denotes its edge set, and ∂_-, ∂_+ denote its respective source and target functions $E_X \to V_X$. The vertex and edge sets of a digraph are assumed to be disjoint and the symbol X used to denote a digraph is identified with the disjoint union $V_X \cup E_X$, regarded as a poset ordered so that $v \leqslant_X e$ if e is an edge having v as either its source or target. Fix a digraph X. For each subset $C \subset X$, let

$$star_C = C \cup \bigcup_{v\in V_X\cap C} \partial_-^{-1}(v) \cup \partial_+^{-1}(v).$$

Let $sd\,X$ denote the digraph such that $V_{sd\,X} = X$, $E_{sd\,X} = E_X \times \{-,+\}$, and

$$\partial_-(e,-) = \partial_-(e), \quad \partial_-(e,+) = \partial_+(e,-) = e, \quad \partial_+(e,+) = e, \quad e \in E_X.$$

For each subset $C \subset X$, let $sd\,C$ denote the subset

$$sd\,C = C \cup \{(e,-) \mid e \in C \cap E_X\} \cup \{(e,+) \mid e \in C \cap E_X\} \subset sd\,X.$$

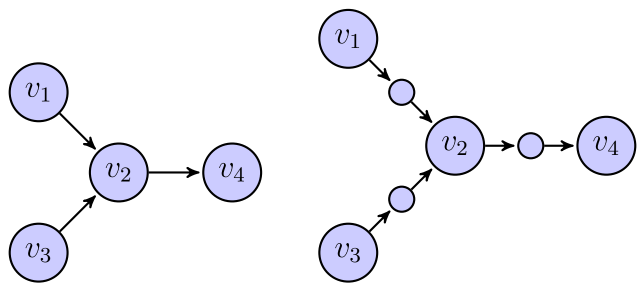


FIGURE 1. **Subdivisions** A digraph X and its subdivision $sd\,X$.

Consider digraphs X and Y. This note writes $X \subset Y$ to indicate that $V_X \subset V_Y$, $E_X \subset E_Y$, and the source and target maps $\partial_-, \partial_+ : E_Y \to V_Y$ are restrictions and corestrictions of respective source and target maps $\partial_-, \partial_+ : E_X \to V_X$.

4. Semimodules

Fix a closed monoidal category $\mathscr{C}$. Commutative monoid objects in $\mathscr{C}$ will generalize ground rings for this note. Listed below are some examples of commutative monoid objects in $\mathscr{C}$.

$\mathscr{C}, \otimes$	**commutative monoid objects in $\mathscr{C}$**
based sets, Cartesian product	commutative monoids
Abelian groups, bilinear tensor	commutative rings
commutative monoids, bilinear tensor	commutative semirings

This note takes a *partial commutative monoid* to mean a commutative monoid object in the Cartesian monoidal category of sets and partial functions between them and a *partial commutative semiring* to mean a commutative monoid object in the category of partial commutative monoids and (partial) homomorphisms between them, equipped with the standard bilinear tensor defined by [...].

Example 4.1. ...

Fix a commutative monoid object S in $\mathscr{C}$ throughout this note. An *S-semimodule* will mean a module object over S and an *S-map* is a morphism of S-semimodules. Let $\mathscr{M}_S$ denote the category of S-semimodules and S-maps between them. Let $S[-]$ denotes the functor from the category of sets and functions to $\mathscr{M}_S$ naturally sending each set X to the X-indexed copower in $\mathscr{M}_S$ of the S-semimodule S. The following proposition follows from [Theorem 2.2, ...].

Proposition 4.2. *There exists a unique tensor product*

$$\otimes : \mathscr{M}_S \times \mathscr{M}_S \to \mathscr{M}_S$$

turning the category $\mathscr{M}_S$ of S-semimodules and S-maps between them into a closed symmetric monoidal category with closed structure $\hom_S$ *sending each pair* (A, B) *of S-semimodules to the $\mathscr{C}$-object* $\hom_{\otimes}(A, B)$ *equipped with ....*

Example 4.3.

An S-semimodule M is *flat* if the functor

$$- \otimes_S M : \mathscr{M}_S \to \mathscr{M}_S$$

preserves finite limits. In the case S is a commutative semiring, 1 will denote the multiplicative unit, 0 will denote the additive identity, and each element $x \in X$ is identified with the image of 1 under the natural inclusion $S \hookrightarrow \bigoplus_{x \in X} S$ mapping S onto the x-indexed summand. A semiring S is *module-free* if the only submodule of S is trivial.

5. Sheaves

Fix a digraph X. An S-*sheaf* on X will mean a functor

$$X \to \mathscr{M}_S$$

from the poset X. The *stalks* of an S-sheaf $\mathcal{F}$ are the S-semimodules $\mathcal{F}(c)$ for each $c \in X$. The *restriction maps* of an S-sheaf are all S-maps of the form $\mathcal{F}(v \leqslant_X e)$ for $v \in V_X$ and $e \in E_X$. The *constant sheaf at an S-semimodule*, written k_S, is the constant sheaf whose restriction maps are the identity on M. An S-sheaf $\mathcal{F}$ on X determines an S-sheaf on $sd\,X$ sending each $v \in V_{sd\,X} = X$ to $\mathcal{F}(v)$, each edge of the form $(e, -)$ or $(e, +)$ to $\mathcal{F}(e)$, and each restriction map to an appropriate restriction map of $\mathcal{F}$ or the identity map between stalks of $\mathcal{F}$. For each inclusion $X \subset Y$ of digraphs and S-sheaf $\mathcal{F}$ on X,

$$(X \subset Y)_* \mathcal{F}$$

denotes the unique S-sheaf on Y such that $(X \subset Y)_*\mathcal{F}(c) = \mathcal{F}(c)$ for each $c \in X$, $(X \subset Y)_*\mathcal{F}(c) = 0$ for each $c \in Y - X$, and $(X \subset Y)_*\mathcal{F}(v \leqslant_X e) = \mathcal{F}(v \leqslant_X e)$ for each $v \leqslant_X e$. This note abuses notation and denotes such a sheaf on $sd\,X$ determined by an S-sheaf $\mathcal{F}$ on X by $\mathcal{F}$.

Example 5.1. ...

Orientation sheaves over rings of weak homology manifolds [2] generalize to *orientation sheaves $\mathcal{O}_S$ over S*, local top dimensional *directed homology* with S-coefficients, for analogues of weak homology manifolds equipped with distinguished directions. For brevity, this note combinatorially constructs $\mathcal{O}_S$; the reader should refer to [**?**] for a principled definition.

Definition 5.2. Let $\mathcal{O}_S, \mathcal{E}_S, \mathcal{V}_S$ be the S-sheaves on X in the diagram

$$\mathcal{O}_S \dashrightarrow \mathcal{E}_S \underset{\partial_+}{\overset{\partial_-}{\rightrightarrows}} \mathcal{V}_S \tag{4}$$

such that $\mathcal{V}_S(e) = 0$ and $\mathcal{E}_S(e) = S[e]$ for $e \in E_X$, $\mathcal{V}_S(v) = S[v]$, and $\mathcal{E}_S(v) = S[\partial_-^{-1}(v) \cup \partial_+^{-1}(v)]$ for $v \in V_X$, and $\mathcal{E}_S(v \leqslant_X e)(c) = 0$ for $c \neq e$ and e for $c = e$ for $v \leqslant_X e$. The above sheaf maps ∂_-, ∂_+ are defined on edges e by $\partial_e(e) = e$ and defined on vertices v by ∂_v restricted to $S[e]$ is the natural identity between copies of S if $\partial e = v$ and the 0-map otherwise, for $\partial = \partial_-, \partial_+$.

Example 5.3. For S a semiring,

$$\mathcal{E}_S(v \leqslant_X e)(c) = \begin{cases} c, & c = e \\ 0. & c \neq e \end{cases}$$

For the case S a ring, $\mathcal{E}_S$ and $\mathcal{V}_S$ are the local $\mathcal{F}$-valued 1-chains and 0-chains on X and $\partial_+ - \partial_-$ is the natural boundary homomorphism. The following proposition follows for the case $M = S$ immediately and for the general case by the Universal Coefficients Theorem for homology.

Proposition 5.4. *Suppose S is a ring. There exists an isomorphism*

$$\mathcal{O}_S(c) \otimes_S M \cong H_1((X, X - star_C); M)$$

natural in cells c in a given digraph and S-modules M, where $H_\bullet$ denotes ordinary simplicial homology.

The local orientations over the natural numbers $\mathbb{N}$ are generated, if not necessarily freely, as local combinatorial directed paths.

Lemma 5.5. *Fix $v \in V_X$. The elements in*

$$(\partial_-^{-1}(v) \cap \partial_+^{-1}(v)) \cup \{e_- + e_+ \mid e_- \in \partial_-^{-1}(v) \setminus \partial_+^{-1}(v),\ e_+ \in \partial_+^{-1}(v) \setminus \partial_-^{-1}(v)\}.$$

individually generate minimal $\mathbb{N}$-subsemimodules of $\mathcal{O}_\mathbb{N}$ and collectively generate all of $\mathcal{O}_\mathbb{N}(v)$.

Proof. Let E_v, E_v^-, E_v^+ be the sets

$$E_v = \partial_-^{-1}v \cap \partial_+^{-1}v, \quad E_v^- = \partial_-^{-1}v \setminus E_v, \quad E_v^+ = \partial_+^{-1}v \setminus E_v.$$

Each $e \in E_v$, indecomposable as an element in $\mathcal{O}_\mathbb{N}(v)$ by e indecomposable as an element in $\mathbb{N}[E_G]$, lies in $\mathcal{O}_\mathbb{N}(v)$ because the parallel arrows both send e to 1.

Consider $e_- \in E_v^-$ and $e_+ \in E_v^+$. Then $e_- + e_+ \in \mathcal{O}_\mathbb{N}(v)$ because both parallel arrows send $e_- + e_+$ to $1 + 0 = 0 + 1 = 1$. Moreover, $e_- + e_+$ is indecomposable because $e_-, e_+ \notin \mathcal{O}_\mathbb{N}(v)$ by $e_-, e_+ \notin E_v$.

Every element in $\mathcal{O}_\mathbb{N}(v)$ factors as a sum of the form

$$\sum_{i \in \mathcal{I}} e_i + \sum_{i \in \mathcal{J}} e_j, \quad e_i \in E_v,\ i \in \mathcal{I} \quad e_j \in E_v^- \cup E_v^+,\ j \in \mathcal{J}. \tag{5}$$

for some indexing sets $\mathcal{I}, \mathcal{J}$. The first sum in (5) is generated by the elements in E_v. Moreover,

$$\#\mathcal{I} + \#\{j \in \mathcal{J} \mid e_j \in E_v^-\} = \partial_-(z) = \partial_+(z) = \#\mathcal{I} + \#\{j \in \mathcal{J} \mid e_j \in E_v^+\},$$

hence $\#\{j \in \mathcal{J} \mid e_j \in E_v^-\} = \#\{j \in \mathcal{J} \mid e_j \in E_v^+\}$, hence $\mathcal{J}$ is the disjoint union of bijective subsets $\mathcal{J}_-, \mathcal{J}_+$ such that $e_j \in E_v^-$ if $j \in \mathcal{J}_-$ and $e_j \in E_v^+$ if $j \in \mathcal{J}_+$. For any choice of bijection $\tau : \mathcal{J}_- \cong \mathcal{J}_+$, the second sum in (5) is generated by elements of the form $e_j + e_{\tau(j)}$ for $j \in \mathcal{I}_-$. □

Lemma 5.6. *Fix $v \in V_X$. Then*

$$(\partial_-^{-1}(v) \cap \partial_+^{-1}(v)) \cup \{e_- + e_+ \mid e_- \in \partial_-^{-1}(v) \setminus \partial_+^{-1}(v),\ e_+ \in \partial_+^{-1}(v) \setminus \partial_-^{-1}(v)\}. \tag{6}$$

freely generates $\mathcal{O}_S(v)$ if v has in-degree or out-degree 1.

Proof. It suffices to consider the case v has in-degree 1, the case v has out-degree 1 symmetrically following. Then there exists a unique $e_- \in \partial_-^{-1}(v)$. Let e_+ denote an element in $\partial_+^{-1}(v)$ and e denote an element of the form e_+ or e_-. The map $(\partial_-)_v : \mathcal{E}_S(v) \to \mathcal{V}_S(v)$ is the isomorphism $S[e_-] \cong S[v]$ sending e_- to v. Hence

$$\mathcal{O}_S(v) = \{\sum_e \lambda_e e \mid \lambda_e \in S,\ \lambda_{e_-} \sum_{e_+} \lambda_{e_+}\} = \{\sum_{e_+} \lambda_{e_+}(e_- + e_+) \mid \lambda_{e_+} \in S\} = S[X]$$

for X the set (6). ☐

Lemma 5.7. *Fix $v \in V_X$. The natural diagram*

$$\mathcal{O}_S(v) \otimes_S M \dashrightarrow \mathcal{E}_S(v) \otimes_S M \underset{\partial_+ \otimes_S M}{\overset{\partial_- \otimes_S M}{\rightrightarrows}} \mathcal{V}_S(v) \otimes_S M, \tag{7}$$

where the dotted arrow is induced by the natural inclusion $\mathcal{O}_S \to \mathcal{E}_S$, is an equalizer diagram natural in S-semimodules M if M is flat, S is a ring, or v has in-degree 1, or v has out-degree 1.

Proof. For M flat, $M \otimes_S -$ sends the equalizer diagram (4) to an equalizer diagram.

For S a ring, the difference between parallel arrows in (7) is the degree 1 differential in the chain complex of local simplicial chains at v with coefficients in M. Hence the equalizer of the solid arrows in (7) is the first local simplicial homology at v with coefficients in M at v. That local homology module naturally is isomorphic to $\mathcal{O}_S(v) \otimes_S M$ [Proposition 5.4].

Consider the case there exists a unique edge $e_- \in E_X$ such that $\partial_- e_- = v$. Let e_+ denote an element in $\partial_+^{-1}(v)$. Then (7) is isomorphic to the diagram

$$\bigoplus_{e_+} M \xrightarrow{\bigoplus_{e_+} \iota_{e_+}} \bigoplus_{e \in \partial_-^{-1}(v) \cup \partial_+^{-1}(v)} M \underset{\partial_+}{\overset{\partial_-}{\rightrightarrows}} M, \tag{8}$$

by Lemma 5.6, where ι_{e_+} is the sum of inclusion of M into the e_+th summand and inclusion of M into the e_-th summand and ∂ maps the eth summand isomorphically onto M if $\partial_- e = v$ and 0 otherwise for $\partial = \partial_-, \partial_+$. The diagram (8) is an equalizer diagram by inspection. ☐

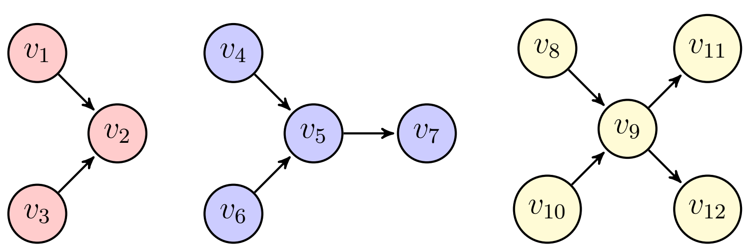


FIGURE 2. **Free and non-free orientations** While $\mathcal{O}_{\mathbb{N}}(v_2) = 0$ and $\mathcal{O}_{\mathbb{N}}(v_5) \cong \mathbb{N} \oplus \mathbb{N}$ are free $\mathbb{N}$-semimodules, $\mathcal{O}_{\mathbb{N}}(v_9)$ is isomorphic to the quotient of $\mathbb{N}[\gamma_1, \gamma_2, \gamma_3, \gamma_4]$ modulo the relation $\gamma_1 + \gamma_2 = \gamma_3 + \gamma_4$ and hence is not a free $\mathbb{N}$-semimodule. However, $\mathcal{O}_{\mathbb{N}}(v_2) = \mathbb{Z}$, $\mathcal{O}_{\mathbb{Z}}(v_5) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $\mathcal{O}_{\mathbb{Z}}(v_9) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ are all free $\mathbb{Z}$-modules.

Orientation sheaves on digraphs measure the degree to which a digraph bifurcates; in other words, orientation sheaves restrict to constant sheaves on directed cycles and directed paths unbounded in the past and future.

Lemma 5.8. *On each digraph, there exist an isomorphism*

$$\mathcal{O}_S \cong k_S$$

if each vertex in the digraph has in-degree and out-degree both 1 *or the semiring* S *is a ring and each vertex in the digraph has total degree* 2*.*

Proof. Consider the case that for each vertex v there exist unique $e_-(v) \in \partial_-^{-1}(v)$ and $e_+(v) \in \partial_+^{-1}(v)$. Then $\mathcal{O}_S(v) = S[e_-(v) + e_+(v)]$ and $\mathcal{O}_S(\partial e \leqslant_X e)$ sends $e_-(\partial e) + e_+(\partial e)$ to $e_-(\partial e)$ or $e_+(\partial e)$ for $\partial = \partial_-, \partial_+$ [Lemma 5.6].

In the case S is a ring and each vertex has total degree 2, $\mathcal{O}_S$ is the orientation sheaf over S on a 1-manifold [Proposition 5.4], which is orientable over S. □

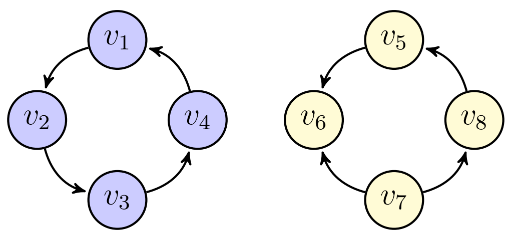


FIGURE 3. **Constant orientations** Over the left digraph, $\mathcal{O}_{\mathbb{N}} = k_{\mathbb{N}}$. Over both digraphs, $\mathcal{O}_{\mathbb{Z}} = k_{\mathbb{Z}}$. Over the right digraph, $\mathcal{O}_{\mathbb{N}}(v_6) = 0$ and hence $\mathcal{O}_{\mathbb{N}} \neq k_{\mathbb{N}}$.

An inclusion $X \subset Y$ of digraphs induces stalkwise inclusions

$$(X \subset Y)_* \mathcal{V}_S \to \mathcal{V}_S, \quad (X \subset Y)_* \mathcal{E}_S \to \mathcal{E}_S,$$

which in turn together induce a stalkwise inclusion

$$(X \subset Y)_* \mathcal{O}_S \to \mathcal{O}_S$$

of S-sheaves on Y.

Definition 5.9. An S-sheaf $\mathcal{F}$ is *locally hard* if for each pair $v \leqslant_X e$,

$$\mathcal{F}(v \leqslant_X e)^{-1}(0) = 0.$$

This note writes $Sh_{X;S} = \langle Sh_{X;S}, \otimes_S, k_S \rangle$ for the closed symmetric monoidal category of cellular sheaves on a digraph X and natural transformations between them, with tensor $\otimes_S$ inherited pointwise from $\mathscr{M}_S$.

6. (Co)homology

This section constructs H^0, H_0, H_1 for sheaves of semimodules on digraphs. For brevity, this note eschews a general construction of directed sheaf (co)homology introduced in [7] and instead combinatorially constructs the theories for the special case of interest.

6.1. $\mathbf{H^0}$. Let $H^0(C;\mathcal{F})$ denote a choice of subobject

$$H^0(C;\mathcal{F}) \subset \lim_{c\in C} \mathcal{F}(c) \tag{9}$$

of the inverse limit $\lim_{c\in C}\mathcal{F}(c)$, natural in subsets $C \subset X$ and S-sheaves $\mathcal{F}$ on X. The functor H^0 is called the *local sections functor* in the case that (9) is an equality.

Example 6.1. Equivalently $H^0(C;\mathcal{F})$ is given by an equalizer diagram

$$H^0(C;\mathcal{F}) \dashrightarrow \prod_{v\in C\cap V_X} \mathcal{F}(v) \underset{\pi_+}{\overset{\pi_-}{\rightrightarrows}} \prod_{e\in C\cap E_X} \mathcal{F}(e),$$

where $(\pi_-(\phi))_e = (\phi_{\partial_- e})_e$ and $(\pi_+(\phi))_e = (\phi_{\partial_+ e})_e$ for each $e \in E_X$, natural in sheaves $\mathcal{F}$ of S-semimodules on digraphs X and subsets $C \subset X$, for H^0 the local sections functor.

Example 6.2. For each S-sheaf $\mathcal{F}$ on a digraph X and $C \subset E_X$,

$$H^0(C;\mathcal{F}) = \prod_{e\in C} \mathcal{F}(e)$$

since C is a disjoint union of singletons as a poset diagram.

Consider $\sigma \in H^0(X;\mathcal{F})$. The *support of* σ, written $|\sigma|$, is the subset

$$|\sigma| = \{c \in X \mid \sigma_c \neq 0\} \subset X.$$

Supports of global sections always form subgraphs because restriction maps preserve additive identities. The *restriction of* σ *to* $c \in X$, written σ_c, is the image of σ under the S-map $H^0(\{c\} \subset X;\mathcal{F})$.

Lemma 6.3. *Suppose S is a semiring. For S-sheaves $\mathcal{F},\mathcal{G}$ on X as in*

$$\begin{array}{ccc} \mathcal{A} & \dashrightarrow & \mathcal{F}\otimes_S \mathcal{G} \\ \vdots & & \\ \downarrow & & \\ \bigoplus_{i\in\mathcal{I}} \mathcal{F} & & \end{array}$$

with $\mathcal{G}$ locally hard, there exist dotted sheaf maps with the dotted vertical map an injection such that for each $C \subset X$, $H^0(C;\epsilon)$ is surjective.

Proof. Let $\mathcal{G}_\flat$ be the S-sheaf on X defined on $c \in X$ by

$$\mathcal{G}_\flat(c) = \bigoplus_{c\in C} S[UH^0(C;\mathcal{G})],$$

where C ranges over subsets of X satisfying $C = \partial_-^{-1}(C)\cup\partial_+^{-1}(C)$ and UM denotes the underlying set of an S-semimodule M, and whose restriction maps are inclusions. Let $\epsilon : \mathcal{G}_\flat \to \mathcal{G}$ be the sheaf map such that ϵ_c is the composite of the counit S-map $S[UH^0(C;\mathcal{G})] \to H^0(C;\mathcal{G})$ of the adjunction $S[-] \vdash U$ with $H^0(c \subset C;\mathcal{G})$, for each $c \in X$. Then $\mathcal{G}_\flat$ is the coproduct of pushforwards of constant sheaves and $H^0(C;\epsilon)$ is surjective for each $C \subset X$ by construction. □

6.2. $\mathbf{H_0}$. Zeroth directed homology classifies stalks up to parallel transport.

Definition 6.4. Let $H_0(C;\mathcal{F})$ be defined by coequalizer diagram

$$H^0(sd\,star_C;\mathcal{F}\otimes_S\mathcal{E}_S)\underset{\partial_+}{\overset{\partial_-}{\rightrightarrows}}H^0(sd\,star_C;\mathcal{F}\otimes_S\mathcal{V}_S)\dashrightarrow H_0(C;\mathcal{F}),$$

natural in S-sheaves $\mathcal{F}$ over a digraph X and subsets $C\subset E_X$.

Thus H_0 dualizes the formulation of H^0 in Example 6.1.

Example 6.5. For each S-sheaf $\mathcal{F}$ on a digraph X and $C\subset E_X$,

$$H_0(C;\mathcal{F})=\prod_{e\in C}\mathcal{F}(e)$$

since C is a disjoint union of singletons as a poset diagram.

Example 6.6 (H_0 as a colimit)**.** For finite subposets $C\subset X$,

$$H_0(C;\mathcal{F})=\operatorname*{colim}_{c\in C}\mathcal{F}(c).$$

Example 6.7. For a connected and finite digraph X,

$$H_0(X;k_S)\cong S.$$

Inclusions $A\subset B\subset X$ induce dotted vertical S-maps of the form

$$\begin{array}{ccccc}
H^0(star_A;\mathcal{V}_S\otimes_S\mathcal{F}) & \underset{\pi_+}{\overset{\pi_-}{\rightrightarrows}} & H^0(star_A;\mathcal{E}_S\otimes_S\mathcal{F}) & \dashrightarrow & H_0(A;\mathcal{F})\\
\downarrow & & \downarrow & & \downarrow{\scriptstyle H_0(A\subset B;\mathcal{F})}\\
H^0(star_B;\mathcal{V}_S\otimes_S\mathcal{F}) & \underset{\pi_+}{\overset{\pi_-}{\rightrightarrows}} & H^0(star_B;\mathcal{E}_S\otimes_S\mathcal{F}) & \dashrightarrow & H_0(B;\mathcal{F}),
\end{array}$$

the left and middle vertical maps induced by projections onto $\mathcal{F}(a)$ for $a\in A$ and the 0-maps to $\mathcal{F}(b)$ for $b\in B-A$, and hence the right vertical map $H_0(A\subset B;\mathcal{F})$ by naturality.

6.3. $\mathbf{H_1}$. First homology is Poincaré dual to cohomology.

Definition 6.8. Let $H_1((X,C);\mathcal{F})$ denote the S-semimodule

$$H_1((X,C);\mathcal{F})=H^0(X-star_C;\mathcal{O}_S\otimes_S\mathcal{F})$$

natural in S-sheaves $\mathcal{F}$ over a digraph X and subsets $C\subset E_X$, with $H_1(X;\mathcal{F})$ short for $H_1((X,\varnothing);\mathcal{F})$.

Proposition 6.9. *For each S-sheaf over X,*

$$H_1(X;\mathcal{F})\cong H^0(X;\mathcal{F})$$

if each vertex has in-degree and out-degree both 1 *or S is a ring and each vertex has total degree* 2.

Proof. Observe that

$$H_1(X;\mathcal{F})=H^0(X;\mathcal{O}_S\otimes_S\mathcal{F})\cong H^0(X;k_S\otimes_S\mathcal{F})\cong H^0(X;\mathcal{F}),$$

the first equality by definition, the middle isomorphism by Lemma 5.8, and the last isomorphism by k_S a unit for $\otimes$ in $Sh_{X;S}$. □

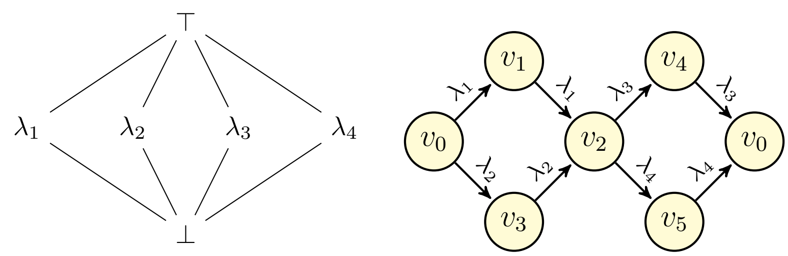


FIGURE 4. **Essential bifurcations** Given the sup-semilattice Λ having Hasse diagram illustrated on the left and digraph X given on the right, the element in the $\mathbb{N}[\Lambda]$-semimodule $H_1(X; k_\Lambda)$ with illustrated restrictions on the right is indecomposable and does not lie in the $\mathbb{N}$-semimodule $H_1(X; k_{\mathbb{N}} \otimes_{\mathbb{N}} k_\Lambda)$, even though $k_\Lambda \cong k_{\mathbb{N}} \otimes_{\mathbb{N}} k_\Lambda$ as $\mathbb{N}$-sheaves.

Inclusions $A \subset B \subset X$ induce S-maps

$$H_1((X,A) \subset (X,B); \mathcal{F}) : H_1((X,A); \mathcal{F}) \xrightarrow{H^0(X-B \subset X-A; \mathcal{O}_S \otimes_S \mathcal{F})} H_1((X.B); \mathcal{F}).$$

Proposition 6.10 (Universal Coefficients)**.** *There exists an S-map*

$$H_1((X,A); \mathcal{F}) \otimes_S M \cong H_1((X,A); \mathcal{F} \otimes_S k_M)$$

natural in S-sheaves $\mathcal{F}$, $A \subset X$, and S-semimodules M and an isomorphism for M flat, where $\mathcal{F} \otimes_S k_M$ is regarded as an S-semimodule.

Proof. There exists a natural cone from $H^0(X - A; \mathcal{O}_S \otimes_S \mathcal{F}) \otimes_S M$ to

$$\prod_{v \in V_{X-A}} \mathcal{O}_S(v) \otimes_S \mathcal{F}(v) \otimes_S M \underset{\partial_+}{\overset{\partial_-}{\rightrightarrows}} \prod_{e \in E_{X-A}} \mathcal{O}_S(e) \otimes_S \mathcal{F}(e) \otimes_S M,$$

inducing a natural map from $H_1((X,A); \mathcal{F}) \otimes_S M$ to the equalizer $H_1(X; \mathcal{F} \otimes_S k_M)$ of the rightmost parallel arrows, an isomorphism for M flat because tensoring by flat semimodules preserves equalizer diagrams. □

Example 6.11 (Necessity of flatness)**.** Observe that

$$H_1(X; k_{\mathbb{N}}) \otimes_{\mathbb{N}} \mathbb{Z} = 0 \ncong H_1(X; k_{\mathbb{N}} \otimes_{\mathbb{N}} k_{\mathbb{Z}}) = H_1(X; k_{\mathbb{Z}}).$$

for X a digraph with no directed loops but at least one undirected cycle. Hence tensoring with $\mathbb{Z}$, not flat as an $\mathbb{N}$

Under either local algebraic or local geometric criteria, $H_1(X; \mathcal{F})$ coincides with a non-Abelian generalization of homology [8] for higher categorical structures; an equalizer condition generalizes the cycle condition and hence such homology semimodules naturally generalize flows. An S-sheaf $\mathcal{F}$ is *flat* if the stalks of $\mathcal{F}$ are flat and *additively invertible* if the stalks of $\mathcal{F}$ are groups. A digraph is *locally finite* if each vertex has finite in-degree and finite out-degree. A (possibly infinite) $\mathcal{I}$-index collection of S-maps $\psi_i : M_i \to N$ for $i \in \mathcal{I}$ induces an S-map

$$\prod_{i \in \mathcal{I}} M_i \to N$$

sending $(m_i)_{i\in\mathcal{I}}$ to the well-defined finite sum $\sum_{\psi_i(m_i)\neq 0}\psi_i(m_i)$ as long as $\psi_i(m_i)\neq 0$ for finitely many $i\in\mathcal{I}$, for each $\mathcal{I}$-indexed tuple $(m_i)_{i\in\mathcal{I}}$ in the domain. In this sense the following theorem holds.

Theorem 6.12. *For a locally finite digraph X, there exists an equalizer diagram*

$$H_1(X;\mathcal{F}) \dashrightarrow \prod_{e\in E_X}\mathcal{F}(\partial_- e)\times_{\mathcal{F}(e)}\mathcal{F}(\partial_+ e) \underset{\pi_+}{\overset{\pi_-}{\rightrightarrows}} \prod_{v\in V_X}\mathcal{F}(v),$$

with π_-,π_+ the maps induced by projections onto first and second factors, for an S-sheaf $\mathcal{F}$ on X such that $\mathcal{F}$ is flat, S is a ring, or S is a semiring and each vertex in X has in-degree 1 or out-degree 1.

Proof. The sheaf $\mathcal{O}_S\otimes_S\mathcal{F}$ equalizes $\partial_-\otimes_S\mathcal{F},\partial_+\otimes_S\mathcal{F}$, edgewise by $\mathcal{V}_S$ trivial on edges and vertexwise by Lemma 5.7. Hence the equalizer of the top row in

$$\begin{array}{ccc} H^0(X;\mathcal{E}_S\otimes_S\mathcal{F}) & \underset{H^0(X;\partial_+\otimes_S\mathcal{F})}{\overset{H^0(X;\partial_-\otimes_S\mathcal{F})}{\rightrightarrows}} & H^0(X;\mathcal{V}_S\otimes_S\mathcal{F}) \\ \alpha \downarrow & & \downarrow \beta \\ \prod_{e\in E_X}\mathcal{F}(\partial_- e)\times_{\mathcal{F}(e)}\mathcal{F}(\partial_+ e) & \underset{\pi_+}{\overset{\pi_-}{\rightrightarrows}} & \prod_{v\in V_X}\mathcal{F}(v), \end{array}$$

is $H_1(X;\mathcal{F})$ by H^0 continuous. It therefore suffices to construct S-maps α,β inducing an isomorphism from the equalizer of the top diagram to the equalizer of the bottom diagram.

Let ϕ denote an element in $H^0(X;\mathcal{E}_S\otimes_S\mathcal{F})$, v denote a vertex in X, e denote an edge in X, e_-,e_+ respectively denote $\partial_- e,\partial_+ e$, and

$$\alpha_e^-(\phi) = \left(\mathcal{E}_S(e_-\leqslant_X e)\otimes_S 1_{\mathcal{F}e_-}\right)(\phi_{e_-})\in\mathcal{F}(e_-)$$
$$\alpha_e^+(\phi) = \left(\mathcal{E}_S(e_+\leqslant_X e)\otimes_S 1_{\mathcal{F}e_+}\right)(\phi_{e_+})\in\mathcal{F}(e_+).$$

Then $\alpha_e(\phi)=(\alpha_e^-(\phi),\alpha_e^+(\phi))\in\mathcal{F}(e_-)\times_{\mathcal{F}(e)}\mathcal{F}(e_+)$ because

$$\begin{aligned}(\alpha_e^-(\phi))_e &= \left(1_{\mathcal{E}_S(e)}\otimes_S\mathcal{F}(e_-\leqslant_X e)\right)\circ\left(\mathcal{E}_S(e_-\leqslant_X e)\otimes_S 1_{\mathcal{F}e_-}\right)(\phi_{e_-})\\ &= \left(\mathcal{E}_S(e_-\leqslant_X e)\otimes_S\mathcal{F}(e_-\leqslant_X e)\right)(\phi)\\ &= \phi_e,\end{aligned}$$

similarly $(\alpha_e^+(\phi))_e=\phi_e$, and hence $(\alpha_e^-(\phi))_e=(\alpha_e^+(\phi))_e$. Hence let

$$\alpha : H^0(X;\mathcal{E}_S\otimes_S\mathcal{F})\to\prod_{e\in E_X}\mathcal{F}(\partial_- e)\times_{\mathcal{F}(e)}\mathcal{F}(\partial_+ e)$$

be the S-map sending ϕ to $\sum_e\alpha_e(\phi)$ and let

$$\beta : H^0(X;\mathcal{V}_S\otimes_S\mathcal{F})\to\prod_{v\in V_X}\mathcal{F}(v)$$

be the isomorphism sending a global section to the product of its restrictions to vertices.

The map α is injective because each ϕ is determined by restrictions of the form

$$\phi_v\in S[\partial_-^{-1}(v)\cup\partial_+^{-1}(v)]\otimes_S\mathcal{F}(v)\cong\bigoplus_e\mathcal{F}(v)\subset\prod_e\mathcal{F}(v),$$

where e denotes an element in $\partial_-^{-1}(v) \cup \partial_+^{-1}(v)$, each of which are in turn determined by their decompositions into summands on the right, which in turn are projections of $\alpha(\phi)_e$ onto their first and second factors for $e \in \partial_-^{-1}(v)$ and $e \in \partial_+^{-1}(v)$.

Let β denote the natural isomorphism defined as the product of restriction maps to stalks.

The maps α, β induce a map of equalizers by the following argument.

Consider ϕ. We first show that $\beta(H^0(X; \partial_- \otimes_S \mathcal{F})(\phi)) = \pi_-(\alpha(\phi))$. It suffices to consider the case $\phi_v = e_v \otimes \lambda_v$ for some choice of $v \in V_X$, $e_v \in E_X \cap \mathcal{E}_S(v)$, and $\lambda_v \in \mathcal{F}(v)$ - such ϕ generate $H^0(X; \mathcal{E}_S \otimes_S \mathcal{F})$. Then

$$\pi_-(\alpha(\phi)) = \sum_e \eta_e^-(\phi) = \sum_e \mathcal{E}_S(e_- \leqslant_X e)(e_v) \otimes \lambda_v = \sum_v \mathcal{E}_S(v \leqslant_X e_v)(e_v) \otimes \lambda_v$$

is λ_v if $v = \partial_- e_v$ and 0 otherwise. And $\phi_- H^0(X; \partial_- \otimes_S \mathcal{F})(\phi)$ is the global section in $H^0(X; \mathcal{V}_S \otimes_S \mathcal{F})$ restricting to $(\partial_-)_v(\phi_v) \otimes_S \lambda_v$ at v and 0 at all other stalks. Hence $\beta(\phi_-)$ is also λ_v if $v = \partial_- e_v$ and 0 otherwise.

Similarly $\beta(H^0(X; \partial_+ \otimes_S \mathcal{F})(\phi)) = \pi_+(\alpha(\phi))$ for each ϕ.

The map of equalizers induced by α, β is injective by α and surjective by the following argument.

Let γ denote an element in the equalizer of the bottom row, γ_e denote is projection onto the e-indexed factor, π_-, π_+ denote projections of pullbacks of the form $\mathcal{F}(e_-) \times_{\mathcal{F}(e)} \mathcal{F}(e_+)$ onto their first and second factors. For each γ, let

$$\hat{\gamma}_v = \sum_{\partial_- e = v} e \otimes \pi_- \gamma_e + \sum_{\partial_+ e = v} e \otimes \pi_+ \gamma_e, \quad \hat{\gamma}_e = (\gamma_e)_e.$$

Then $\hat{\gamma}$ defines a preimage for γ under α. □

In other words, first directed sheaf homology $H_1(X; \mathcal{F})$ corresponds to a natural homology theory on the *cellular cosheaf* on X defined by pulling back $\mathcal{F}$ along closed cells.

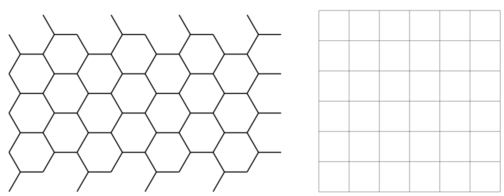


FIGURE 5. **Degree bounds on the vertices** Consider the two graphs above with directionality flowing from left to right and bottom to top. The left side satisfies the degree bounds in the hypothesis of the theorem, while the right side does not.

Corollary 6.13. *Consider the case S a ring. Then the S-module*

$$H_1(X; \mathcal{F})$$

naturally is isomorphic to the first Borel-Moore homology of X with coefficients in an S-sheaf $\mathcal{F}$ on X.

Example 6.14. There exist a dotted S-map making

$$H_1^c(X;k_M) \dashrightarrow S[E_X]\otimes_S M \underset{\pi_+}{\overset{\pi_-}{\rightrightarrows}} S[V_X]\otimes_S M,$$

an equalizer diagram natural in flat S-semimodules M by Theorem 6.12. In the case S a ring, $H_1^c(X;k_M) = ker(\pi_- - \pi_+)$ and hence $H_1^c(X;k_M)$ is the ordinary simplicial homology of X with coefficients in the S-module M.

6.4. **Exactness.** Ordinary sheaf homology is exact. Directed homology comes equipped with connecting homomorphisms from degree 1 to degree 0, although the natural analogue of exactness in the semimodule-theoretic setting fails in general.

Definition 6.15. Let ∂_-,∂_+ denote the S-maps

$$\partial_-,\partial_+ : H_1((X,C);\mathcal{F}) \to H_0(C,\mathcal{F})$$

sending a global section ϕ to the respective representatives of $\prod_{c\in C}(\phi_{\partial_- c})_c$ and $\prod_{c\in C}(\phi_{\partial_+ c})_c$ in $H_0(C;\mathcal{F})$, for each $C\subset E_X$.

Proposition 6.16. *For an S-sheaf $\mathcal{F}$ on X and $C\subset E_X$,*

$$H_1((X,C);\mathcal{F}) \underset{\partial_+}{\overset{\partial_-}{\rightrightarrows}} H_0(C,\mathcal{F}) \xrightarrow{H_0(C\subset X;\mathcal{F})} H_0(X;\mathcal{F})$$

commutes.

Proof. The diagram

$$\begin{array}{ccc}
H^0(X-C;\mathcal{E}_S\otimes_S\mathcal{F}) & \underset{\partial_+}{\overset{\partial_-}{\rightrightarrows}} & H^0(X-C;\mathcal{V}_S\otimes_S\mathcal{F}) \\
\downarrow & & \downarrow \\
H^0(X;\mathcal{E}_S\otimes_S\mathcal{F}) & \underset{\partial_+}{\overset{\partial_-}{\rightrightarrows}} & H^0(X;\mathcal{V}_S\otimes_S\mathcal{F}),
\end{array}$$

where the vertical arrows are extensions by zero, commutes. There exists a natural cone from $H_1((X,C);\mathcal{F})$ to the top pair of arrows. The coequalizer of the bottom row is $H_0(X;\mathcal{F})$. □

Proposition 6.17. *Let S be a ring. For an S-sheaf $\mathcal{F}$ on X and $C\subset E_X$,*

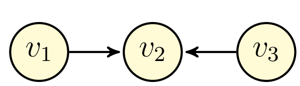
$$\partial_+ - \partial_- : H_1((X,C);\mathcal{F}) \to H_0(C,\mathcal{F})$$

is the ordinary connecting homomorphism for Abelian sheaf homology.

Example 6.18 (Failure of exactness)**.** The commutative diagram

$$H_1((X,C);\mathcal{F}) \underset{\partial_+}{\overset{\partial_-}{\rightrightarrows}} H_0(C,\mathcal{F}) \xrightarrow{H_0(C\subset X;\mathcal{F})} H_0(X;\mathcal{F})$$

is not a coequalizer diagram for X the digraph illustrated below and $C=\{v_1,v_3\}$.



7. A flow-cut duality

This section generalizes the theory of flows and cuts on digraphs in both an algebraic and topological manner.

7.1. Constraints.

Classical constraints on network dynamics often take the form of edge weights on a graph. This note takes an *M-weighted digraph* $(X;\omega)$ to mean a digraph X equipped with E_X-indexed set $\{\omega_e\}_{e\in E_X}$ such that $\omega_e \in M$ for each edge $e \in E_X$, for each set M.

Example 7.1 (Numerical)**.** The ideal

$$\omega_e + \mathbb{Z}^+ = \{\omega_e + 1, \omega_e + 2, \ldots\} = \{x \in \mathbb{N} \mid x > \omega_e\}$$

in the semigroup $\mathbb{N}$ of natural numbers naturally describes all possible forbidden quantities of cars on the road e of a network described by an $\mathbb{N}$-weighted digraph $(X;\omega)$.

Constraints of interest in logistics include multiple commodities on a supply chain subject to bounds on the ratio of their quantities.

Example 7.2 (Multicommodities)**.** The ideal

$$\{v \in \mathbb{R}^n \mid v \cdot c \leqslant \omega_e\} \subset \mathbb{R}^{\geqslant 0} \oplus \mathbb{R}^{\geqslant 0}$$

describes all possible forbidden ratios of two commodities in a supply chain described by an $\mathbb{R}^{\geqslant 0}$-weighted digraph $(X;\omega)$ and vector $c \in \mathbb{R}^n$.

In each of the last three examples, local constraints implicitly define an *spec*(S)-weighted digraph for suitable choices of S. Constraints of interest in information processing [[6], Example 7.3], typically exhibit more interesting restriction maps between the stalks than mere quotients.

Example 7.3 (Information Processing)**.** Let Λ be the *Boolean semiring*

$$\Lambda = \{\top, \bot\}, \quad +_\Lambda = \vee,\ \times_\Lambda = \wedge$$

Free Λ-semimodules encode the possible values of bit-strings and Λ-maps encode logical operations on bit-strings. Hence a stalkwise free Λ-sheaf on a digraph encodes the local functionality of a microprocessor with logical processors at the nodes and local channel bandwidths determined by the size of generating sets for the edge stalks.

Thus S-sheaves on digraphs abstract a range of local constraints on global network states.

7.2. Flows.

Classical flows on a digraph straightforwardly generalize from the setting of real numbers. Consider a partially ordered commutative monoid M. A classical *flow* on a locally finite M-weighted digraph $(X;\omega)$ is a function

$$\phi : E_X \to M$$

satisfying the following conservation law and capacity constraints:

[CONSERVATION] For $v \in V_X$, $\sum_{e\in\partial_-^{-1}(v)} \phi(e) = \sum_{e\in\partial_+^{-1}(v)}$.

[CONSTRAINTS] For $e \in E_X$, $\phi(e) \leqslant_M \omega_e$.

The *e-value* of a classical flow ϕ on $(X;\omega)$ is $\phi(e)$. Classical flows naturally generalize to the sheaf-theoretic setting. Fix an S-sheaf $\mathcal{F}$ on a locally finite digraph X. An *$\mathcal{F}$-valued flow* is an element in the equalizer of the diagram

$$\prod_{e\in E_X} \mathcal{F}(\partial_- e)\times_{\mathcal{F}(e)} \mathcal{F}(\partial_+ e) \underset{\pi_+}{\overset{\pi_-}{\rightrightarrows}} \prod_{v\in V_X}\mathcal{F}(v).$$

The sheaf itself generalizes the capacity constraint. The equalizer condition generalizes the conservation law for classical flows.

Proposition 7.4. *Fix a flat S-semimodule M. Then*

$$H_1(X;\mathcal{F})$$

is isomorphic as a partial $\mathbb{N}$-semimodule to the set of flows on an M-weighted digraph $(X;\omega)$, where $\mathcal{F}$ is the S-sheaf on X assigning $[0,\omega_e]$ to each $e\in E_X$, M to each $v\in V_X$, and injective partial inclusions to each relation $v\leqslant_X e$.

Proof. For each $e\in E_X$,

$$H^0(\{e,\partial_- e,\partial_+ e\};\mathcal{F}) = [0,\omega_e]$$

and for each relation $v\leqslant_X e$, the map $H^0(\{\partial e\}\subset\{e,\partial_- e,\partial_+ e\};\mathcal{F})$ is just inclusion

$$[0,\omega_e]\hookrightarrow M$$

for $\partial=\partial_-,\partial_+$. The claim then follows from Theorem **??**. □

Example 7.5. For a circuit described as a digraph X equipped with an Λ_2-sheaf $\mathcal{F}$ as in Example 7.3, the elements in $H_1(X;\mathcal{F})$ describe the asynchronous executions of the circuit.

The *A-value of a local $\mathcal{F}$-flow over B* is the element in $H_0(X;\mathcal{F})$ represented by $(\phi_a)_a$. This note mimics classical notation $[-:-]$ for flow-values and cut-values from the setting of edge weights to sheaves.

Definition 7.6. For each lattice-ordered S-sheaf $\mathcal{F}$ on X, let

$$[A:B]_{\mathcal{F}}$$

denote the supremum of all A-values of local $\mathcal{F}$-flows over B on X.

Proposition 7.7 (Values)**.** *The dotted map making*

$$\begin{array}{ccc}
H_1(B;\mathcal{F}) & \dashrightarrow & H_0(X;\mathcal{F}) \\
\Big\downarrow{\scriptstyle H_1((B,\varnothing)\subset(B,A);\mathcal{F})} & & \Big\uparrow{\scriptstyle H_0(A\subset B;\mathcal{F})} \\
H_1((B,A);\mathcal{F}) & \xrightarrow[\partial_-]{} & H_0(A;\mathcal{F}).
\end{array}$$

commute sends a local $\mathcal{F}$-valued flow over B to its A-value, for all $A\subset B\subset X$ and $\mathcal{F}$ a flat S-sheaf on X.

7.3. **Cuts.** A (directed) *e-cut* C of X is a subset

$$C \subset V_X \cup E_X$$

such that every (directed) loop ϕ in X traverses e traverses some vertex or cell in C.

Proposition 7.8 (Cuts)**.** *Suppose S is ring-free. Fix $e \in E_X$ and $C \subset X - e$. In*

$$\begin{array}{ccc} H^0(X;\mathcal{O}_S) & \xrightarrow{H^0(C\subset X;\mathcal{O}_S)} & H^0(C;\mathcal{O}_S) \\ {\scriptstyle H^0(e\subset X;\mathcal{O}_S)}\downarrow & & \downarrow{\scriptstyle H_0(X-C\subset X;k_S)\circ\partial_-} \\ H^0(e;\mathcal{O}_S) & \xrightarrow[\partial_-\oplus H_0(e\subset X;k_S)\circ\partial_-]{} & H_0(X;k_S), \end{array}$$

the composite of the top horizontal with the right vertical arrow bounds the composite of the left vertical with the bottom horizontal arrows from above. The diagram commutes if and only if C is an e-cut.

Proof. Let ϕ denote an S-valued flow. Let $\phi_{C,e}$ denote the image of ϕ under

$$H_1((X,\varnothing) \to (X,C);k_S) : H_1(X;k_S) \to H_1((X,X-C);k_S).$$

Let $\bar{\phi}$ denote the images of ϕ under the composite of the left vertical arrow and the bottom horizontal arrow.

Suppose the diagram commutes. In the case $\bar{\phi} \neq 0$, then $\phi_C \neq 0$ by the diagram commutative. In the case $\phi = 0$, then $\phi_{C,e} = -\partial_-\phi_e$, hence $\phi_{C,e} \neq 0$ by $\partial_-\phi_e \neq 0$, hence $\phi_{X-C} \neq 0$, and hence $\phi_C \neq 0$. Hence for each ϕ there exists $c \in C$ such that $\phi_c \neq 0$ in both cases. Thus C is an e-cut over S.

Now suppose C is an e-cut over S. Fix ϕ. It suffices to show that the images of ϕ under both possible composites in the diagram coincide. It therefore suffices to prove the stronger claim that ϕ_C factors into a sum $\phi'_C + \phi''_C$ with $\partial_-\phi'_C = \partial_- e$ and $\partial_-\phi_{C,e} = \phi''_C$ by induction on the minimum length n of an undirected path from C to e in the poset of all e-cuts ordered by inclusion.

In the base case $n = 0$, $C = \{e\}$ and hence $\phi_C = \phi_C + 0$ and $\partial_-\phi_e = \partial_-\phi_C$ and $\partial_-\phi_{C,e} = \partial_- 0$ by $\phi_{C,e} = 0$.

Consider a positive integer $k > 0$, inductively assume the desired factorization holds for the case $n < k$, and now suppose $n = k$. Consider an e-cut B such that the cut-distance from e to B is $n-1$ and there exist distinct $b \in B$ and $c \in C$ with $B - b = B \cap C = C - c$. There exist ϕ'_B, ϕ''_B such that $\phi_B = \phi'_B + \phi''_B$ and $\phi_e = \phi'_B$ and $\phi_{B,e} = \phi''_B$.

Consider the case $\partial_- c = b$. Let ϕ'_C be the image of ϕ'_B under the natural map $H^0(B) \to H^0(C)$. Let ϕ''_C be defined as $\sum_{c \in B\cap C}(\phi''_B)_c$. Then $\partial_-\phi'_C = \partial_-\phi'_B = e$ and $\partial_-\phi''_C = \partial_-\phi_{C,e}$ because $\phi_{C,e}$ consists of all S-valued flows from a cell in C to another cell in C, which is contained in the set of all S-valued flows from a cell in B to a cell in B.

The case $\partial_+ c = b$ similarly follows.

Consider the case $\partial_- b = c$. Let ϕ'_C be the image of ϕ'_B under the natural map $H^0(B) \to H^0(C)$. Let $\phi''_C = \phi''_B + \gamma$, where γ denotes an S-valued flow from C to itself not crossing B. Then $\phi_C = \phi'_C + \phi''_C$ by flow conservation.

The case $\partial_+ b = c$ similarly follows. □

7.4. **Sheaf-theoretic MFMC.** A duality between the values of $\mathcal{F}$-flows and the $\mathcal{F}$-values of cuts evokes and ultimately generalizes MFMC.

Theorem 7.9 (MFMC)**.** *Suppose S is ring-free. For each $e \in E_X$,*

(10) $$[e : X]_{\mathcal{F}} \cong \inf{}_C [C : C]_{\mathcal{F}},$$

where C ranges over all e-cuts of X, for each hard and flat S-sheaf $\mathcal{F}$ on X.

Proof. For brevity, let $H^0(-)$ and $H_\bullet(-)$ denote the constructions

$$H^0(-) = H^0(-; \mathcal{O}_S \otimes_S \mathcal{F}), \quad H_\bullet(-) = H_\bullet(-;\mathcal{F}).$$

There exists a natural isomorphism $H_1(X, e) \cong \lim_C H_1(X, e) \oplus H_1(X - e, C)$ because $\lim_C H_1(X - e, C) = 0$ and inverse limits commute with coproducts. Hence $[e : X]_{\mathcal{F}}$, the image of $H_1((X, e); \mathcal{F})$ under the composite $H_0(e \subset X; \mathcal{F}) \circ \partial_-$, is the well-defined composite π of the natural map $H_1 X \to \lim_C H^0 C$ follows by $H_0(X - C \subset X) \circ \partial_-$ for some choice of C by

(11)
$$\begin{array}{ccc}
H_1(X) & \xrightarrow{H_1(C \subset X)} & \lim_C H^0(C) \\
\downarrow {\scriptstyle H_1((X,\varnothing)\subset(X,e)) \times H_1((X,\varnothing)\to(X,C))} & & \downarrow {\scriptstyle H_0(X - C \subset X)\circ\partial_-} \\
\lim_C H_1(X, e) \oplus H_1(X - e, C) & \xrightarrow[H_0(X - e \subset X)\circ\partial_-]{} & H_0(X)
\end{array}$$

commutative [Proposition 7.8]. The top horizontal map is an isomorphism and hence surjective.

It suffices to show that the vertical map is surjective. For brevity, let C, D denote e-cuts and $(\phi_C)^D$ denote the image of ϕ_C under the lower adjoint of the restriction map $H^0 D \to H^0 C$, for each $\phi_C \in H^0 C$. Let

$$\eta : \prod_C H_0 C \to \prod_C H_0 C$$

be defined by the rule $\eta(\phi)_D = \inf_{C \subset V_X} (\phi_D)_C \wedge \phi_D$ for and $\eta(\phi)_D = \inf_{C \subset E_X} (\phi_D)^C \wedge \phi_D$. Then η is a monotone and decreasing map of complete lattices and hence admits a maximum fixed point ϕ^* which lies in $\lim_C H^0 C$. Moreover, $\inf_C[\phi_C] = \inf_C[(\eta\phi)_C]$ for each ϕ. Hence $\inf_C[\phi^*_C] = [e : X]_{\mathcal{F}}$. □

Example 7.10 (Necessity of locally hardness)**.**

A special case of the theorem is a decomposition of the feasible flow-values as an intersection of all possible local flow-values over cut-sets.

Corollary 7.11. *There exists an isomorphism*

(12) $$[e : X]_{spec\,\mathcal{F}} \cong \bigcap_C [C : C]_{spec\,\mathcal{F}},$$

where C ranges over all e-cuts of X, for each hard but flat S-sheaf and $e \in E_X$.

Corollary 7.12. *For an M-weighted digraph $(X; \omega)$ with edge e_0,*

$$\sup{}_\phi \phi(e_0) = \inf{}_C \sum_{e \in C} \omega_e,$$

where ϕ denotes an M-valued flow ϕ on $(X; \omega)$ and C denotes an e_0-cut.

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