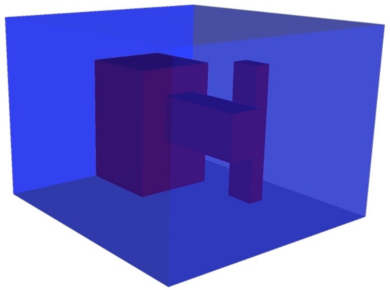
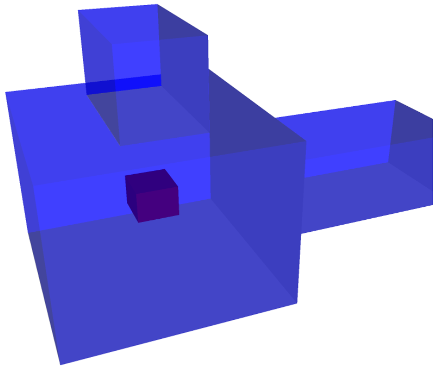
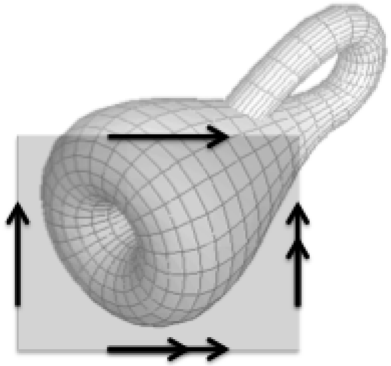
cubical directed homotopy

Sanjeevi Krishnan

CMU HoTT Seminar 03/01/24

I. homotopy theories of categories

category theory

$$F : \mathcal{X} \leftrightarrows \mathcal{Y} : G \quad GF \cong 1_{\mathcal{X}} \quad FG \cong 1_{\mathcal{Y}}$$

THM There exists a model structure on Cat in which …

1. …. weak equivalences are the categorical equivalences
2. …. cofibrations are those functors that are injective on objects

[Joyal and Tierney, 1991]

classical homotopy

$$F : \mathcal{X} \to \mathcal{Y} \qquad BF : B\mathcal{X} \simeq B\mathcal{Y}$$

THM There exists a model structure on Cat whose weak equivalences are those functors that induce classical weak equivalences between nerves

[Thomason, 1991]

directed homotopy

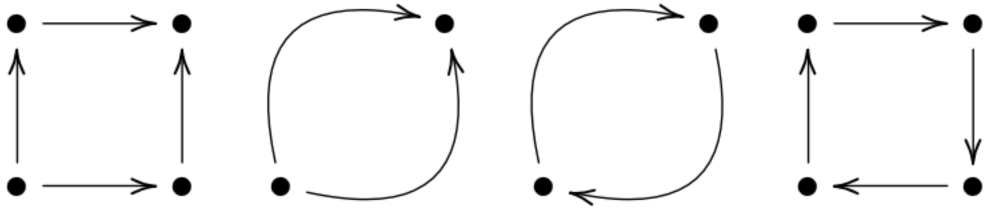
$$F : \mathcal{X} \leftrightarrows \mathcal{Y} : G \qquad FG \to \cdots \leftarrow 1_{\mathcal{X}} \qquad GF \to \cdots \leftarrow 1_{\mathcal{Y}}$$

DEF A directed equivalence between small categories is a functor which is an isomorphism up to zig-zags of natural transformations.

PROP Cat is a **Λ-cofibration** category.

[Minian, 2002]

comparison of homotopy theories

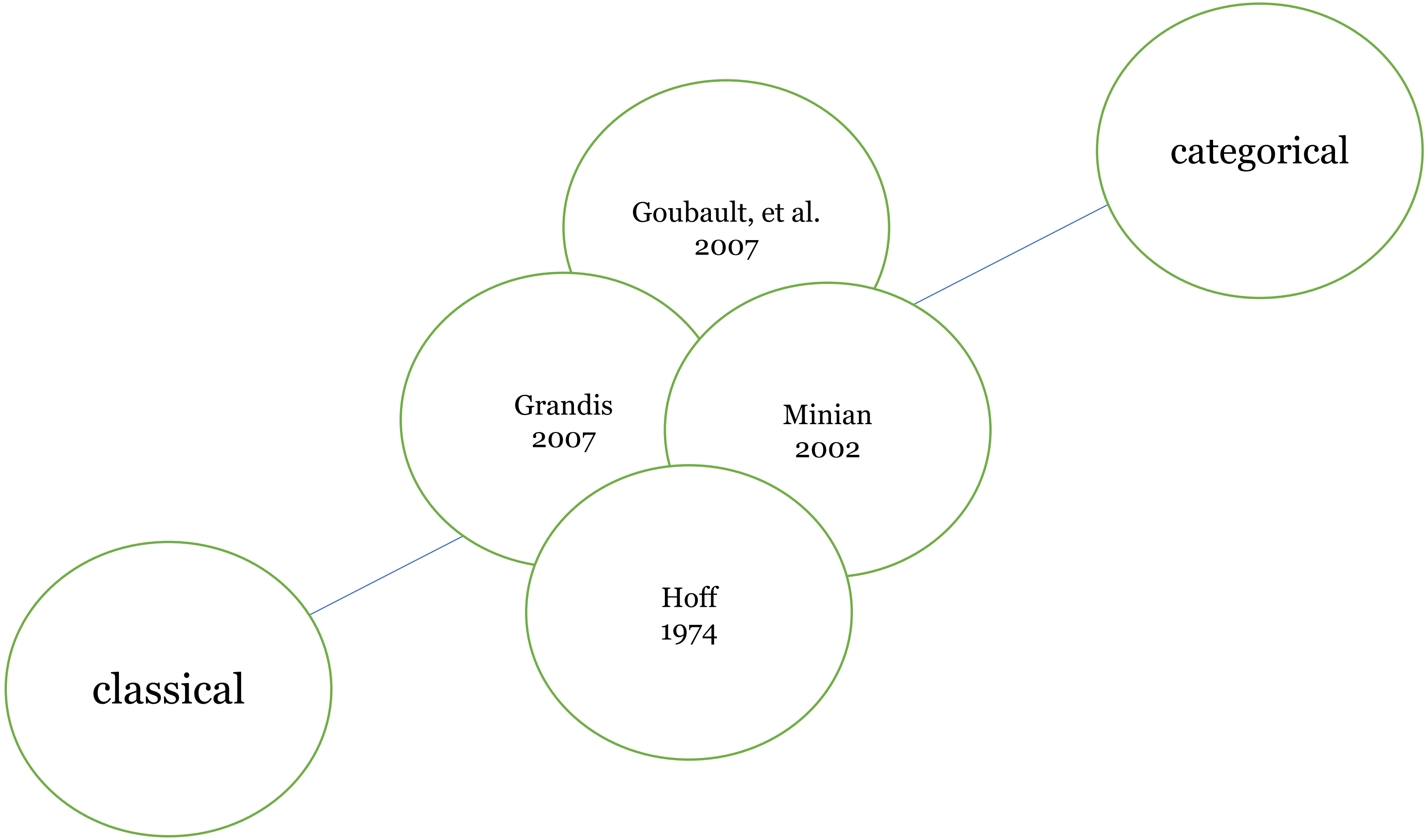


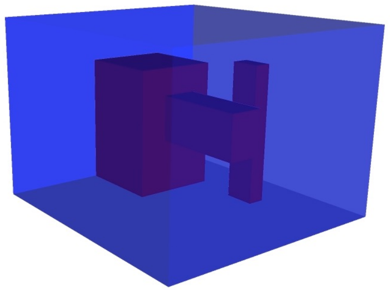
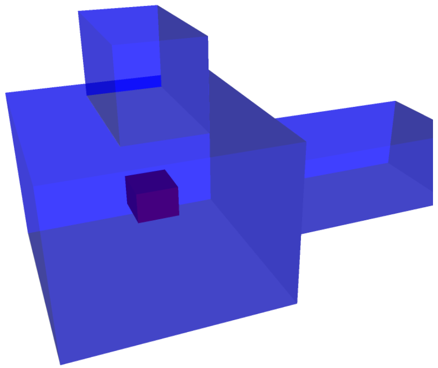
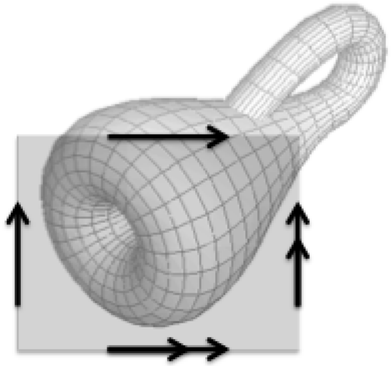
groupoid/Thomason type

directed types

categorical types

an island of technical difficulties

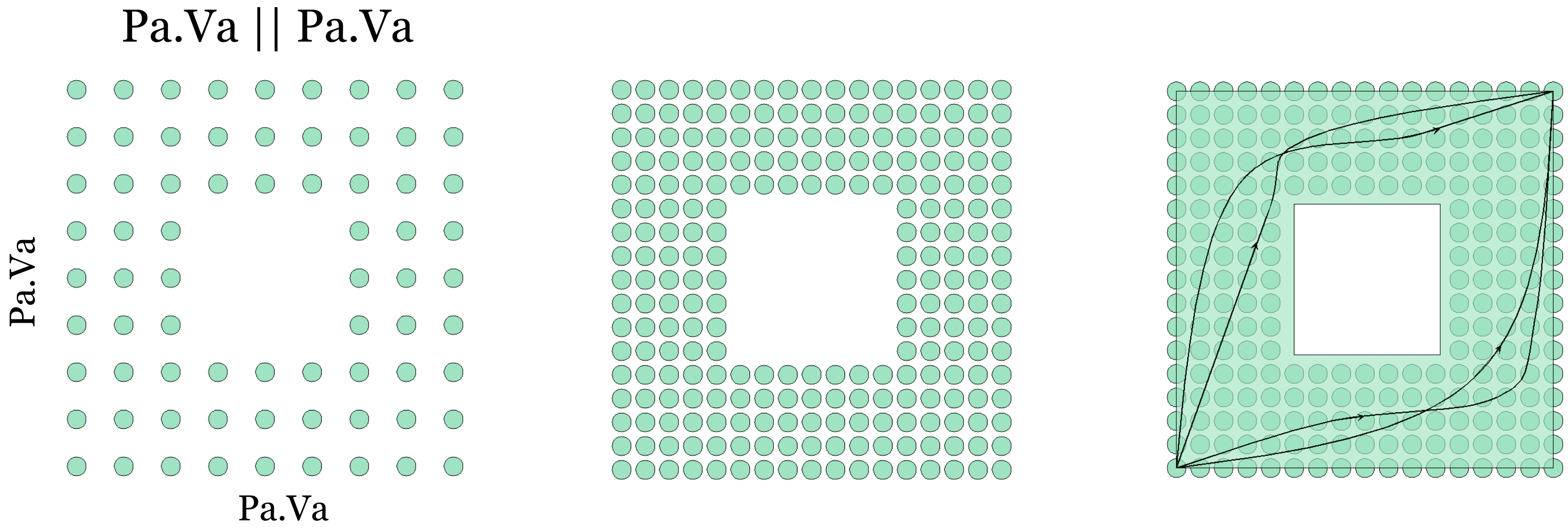


II. directed topology

subdivision invariant state spaces

$$
\begin{aligned}
\langle \text{expr} \rangle ::= {} & P.\langle \text{var} \rangle \\
= {} & V.\langle \text{var} \rangle \\
| \ & \langle \text{expr} \rangle . \langle \text{expr} \rangle \\
| \ & \langle \text{expr} \rangle \parallel \langle \text{expr} \rangle
\end{aligned}
$$



unpredictable behavior

40

START

LET X=4

LET X=X+4

PRINT X

START

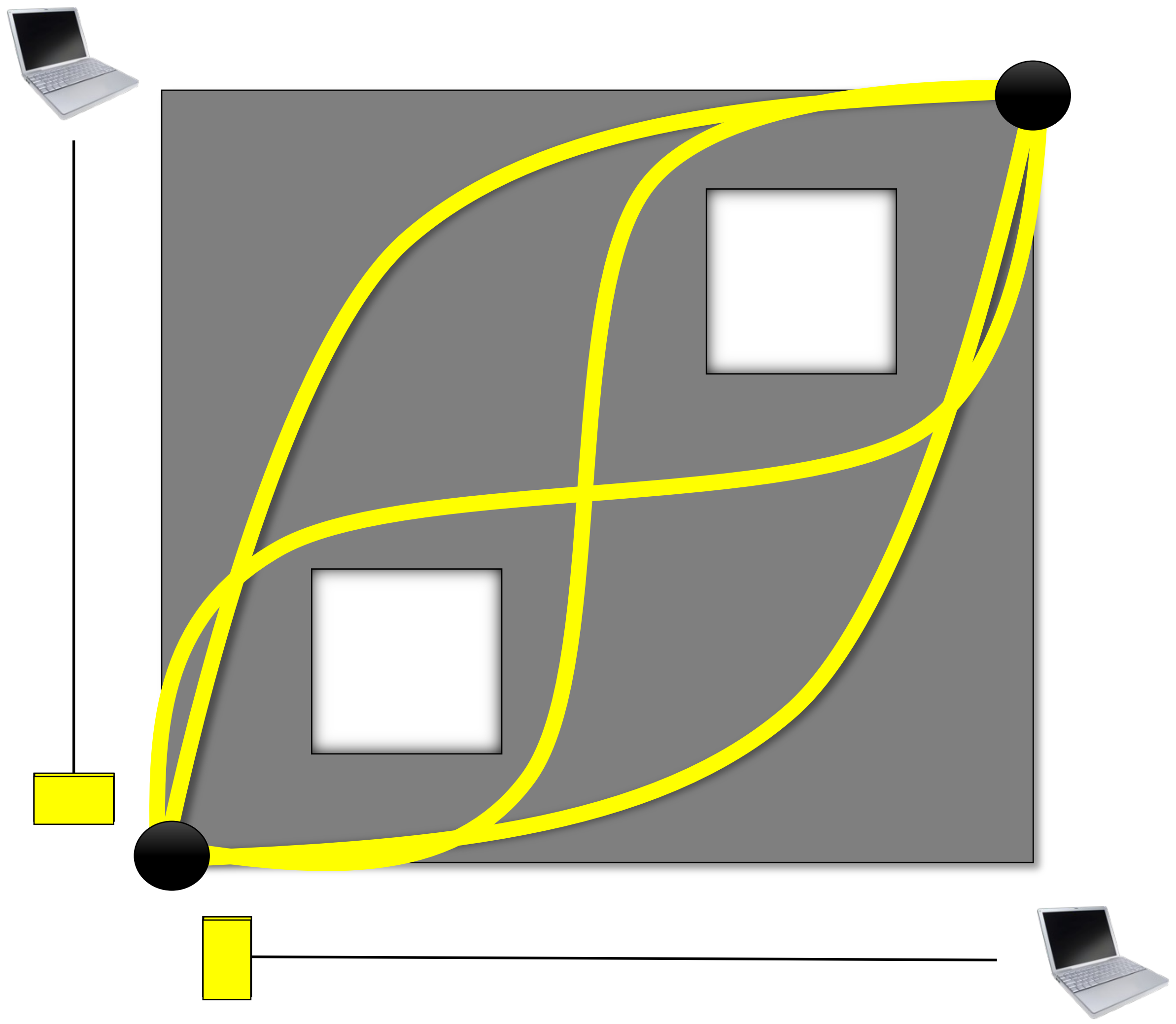
LET X=1

LET X=5X

PRINT X

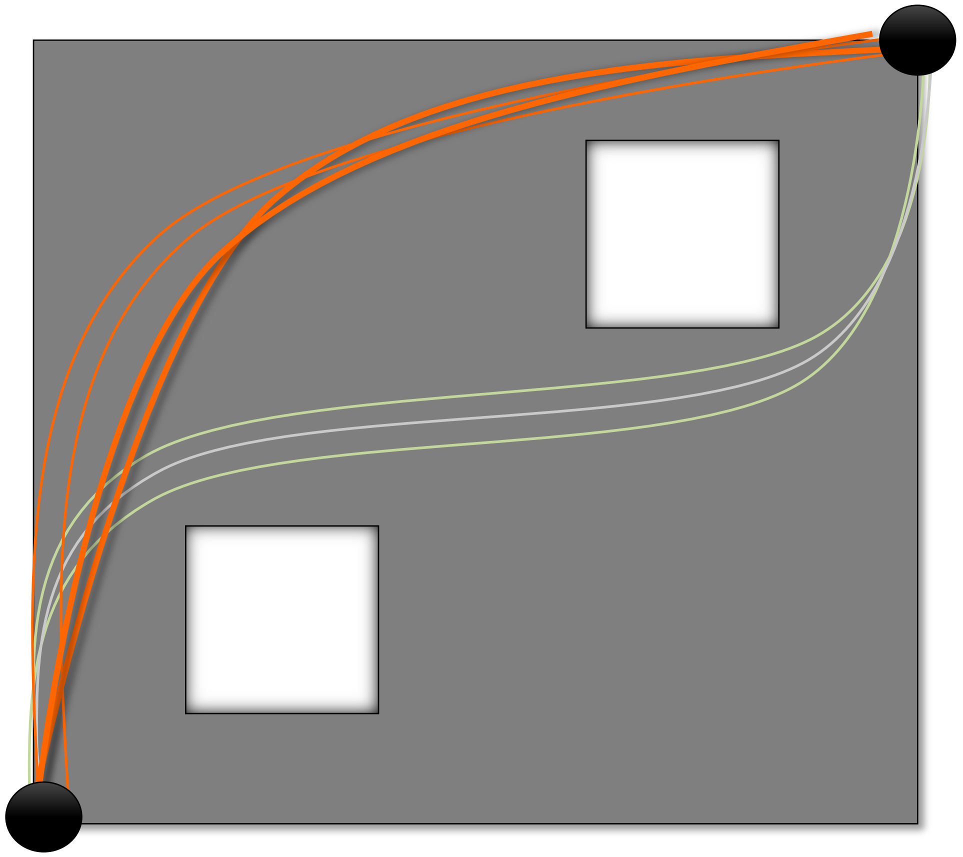
directed paths as executions

stream maps from [0,1]



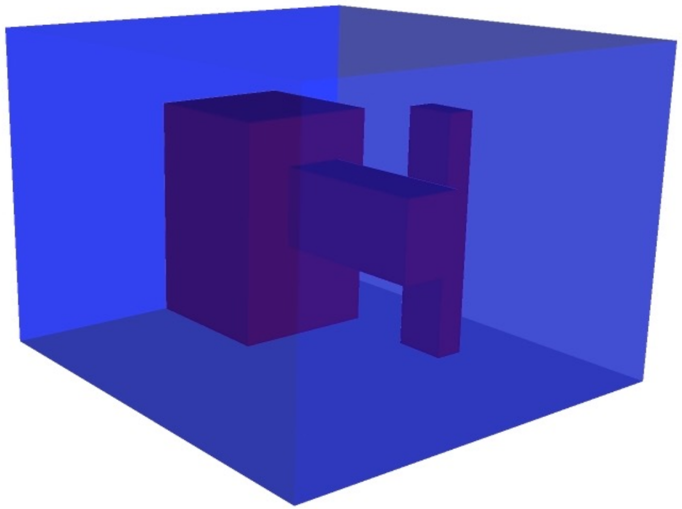
behavior up to homotopy

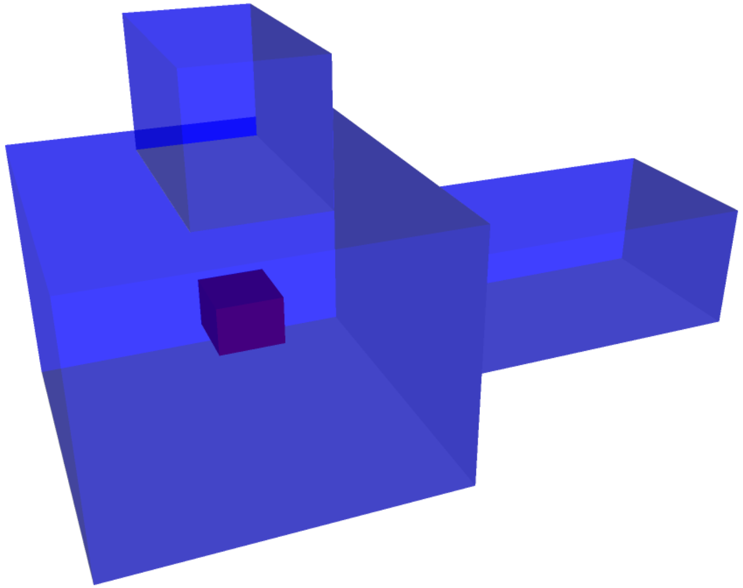
homotopies through paths in time



Paths homotopic through paths in time relative endpoints represent qualitatively equivalent behavior.

more state spaces





some state spaces of 3 concurrent processes

streams

A *stream* is a space X equipped with a function

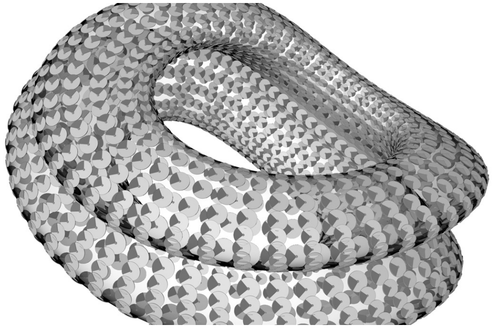
$$U \mapsto \leqslant_U$$

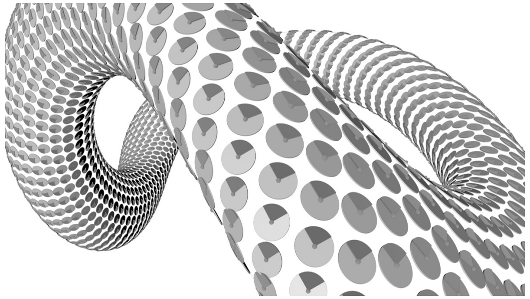
assigning to each open subset U a preorder $\leqslant_U$ on U such that the assignment sends unions to transitive closures of unions.

[Krishnan, 2015]

streams in nature

spacetime: open normal U gets assigned the causal order





streams in nature

THM Every compact Hausdorff connected topological lattice X has the unique structure of a stream in which the preorder on X is the lattice order on X.

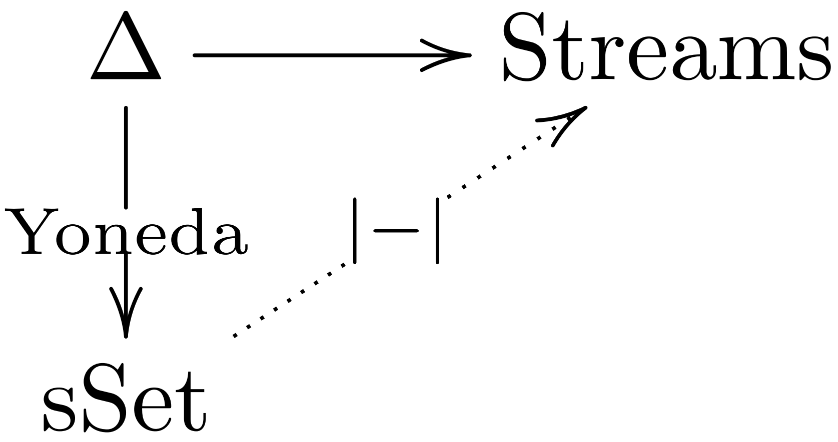
EG topological simplices with lattice operations

$$|\mathrm{ner}(\min : [n]^2 \to [n])|, |\mathrm{ner}(\max : [n]^2 \to [n])| : |\Delta[n]|^2 \to |\Delta[n]|$$

EG topological cubes with product orders

[Krishnan, 2015]

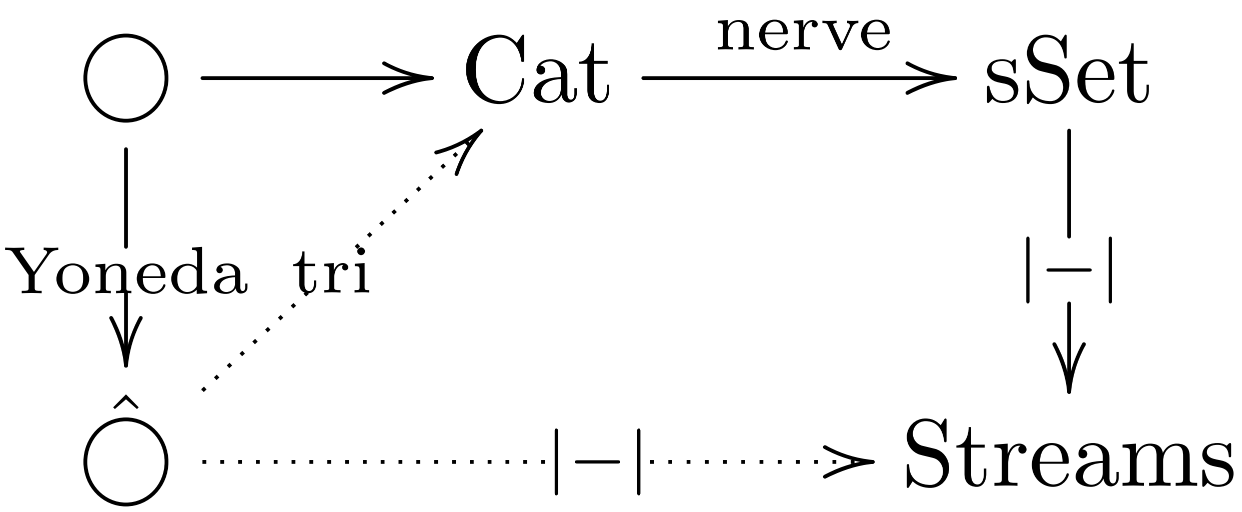
stream realizations



stream realization functor: left Kan extension

[Krishnan, 2015]

stream realizations



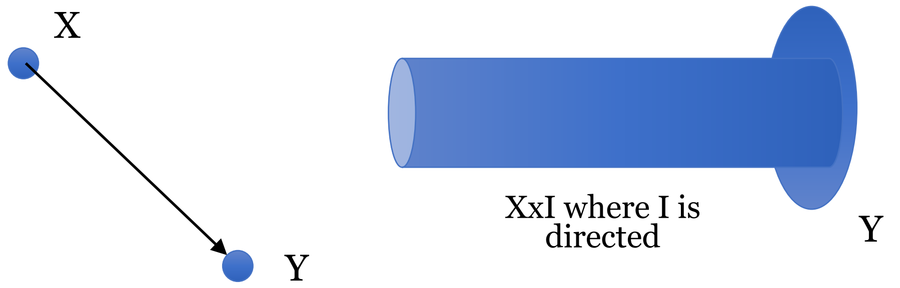
stream realization extends to presheaves over a small subcategory $\bigcirc \subset \mathrm{Cat}$

[Krishnan, 2015]

streams in nature

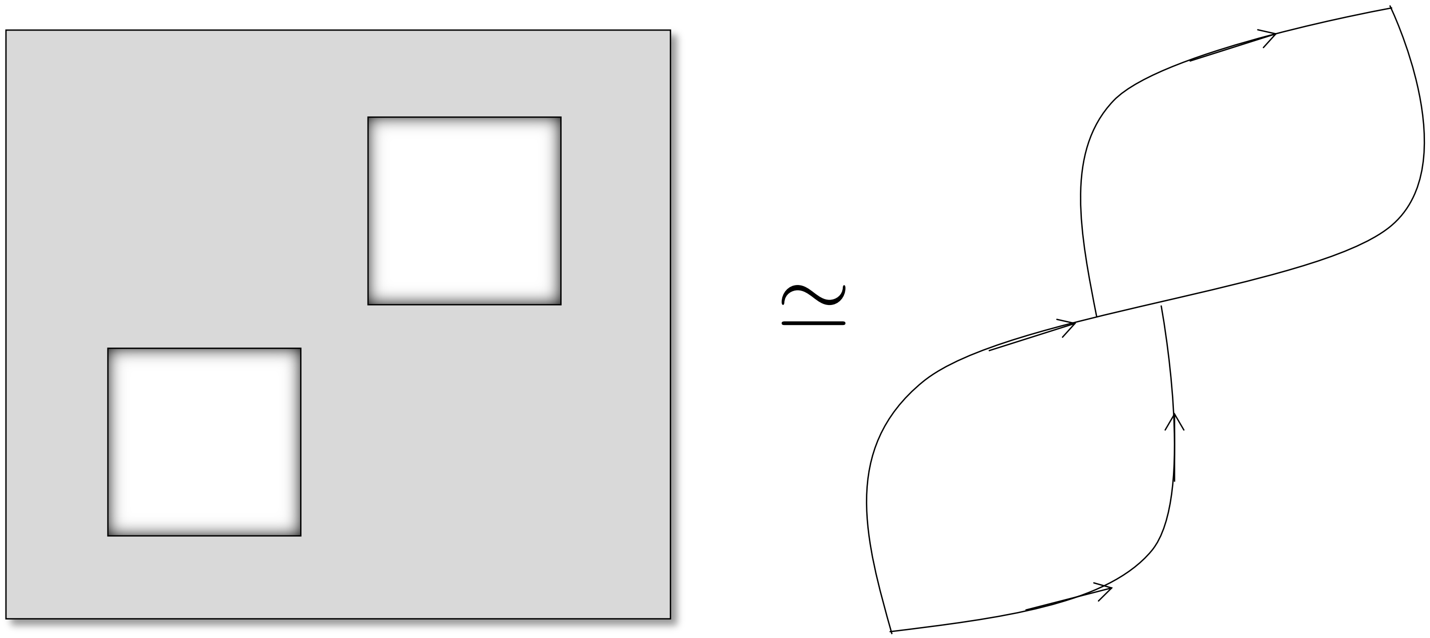
EG stream realizations of nerves of small categories

EG more general homotopy colimits of dynamical systems



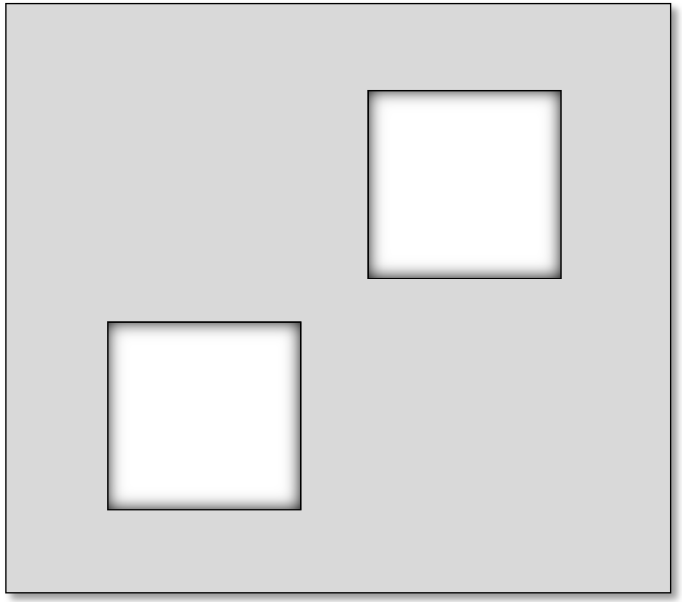
directed homotopy equivalence

dihomotopy equivalence = stream map with inverse up to **directed homotopy**

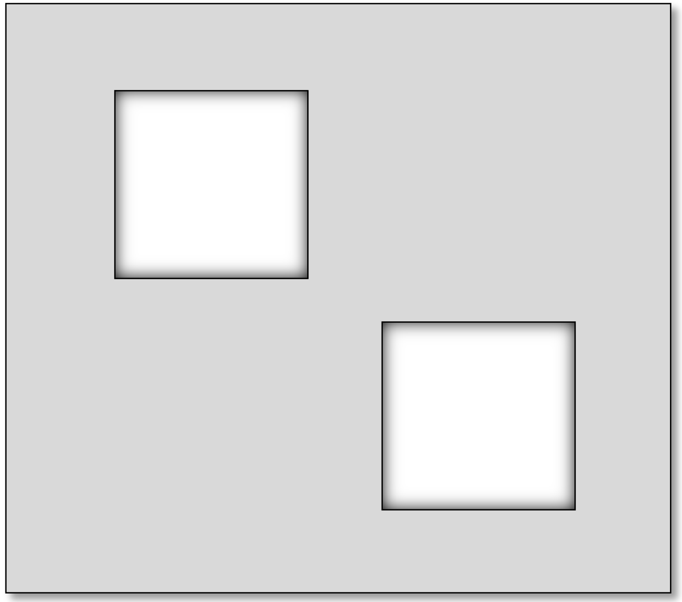


state spaces of systems with similar qualitative features

directed homotopy types

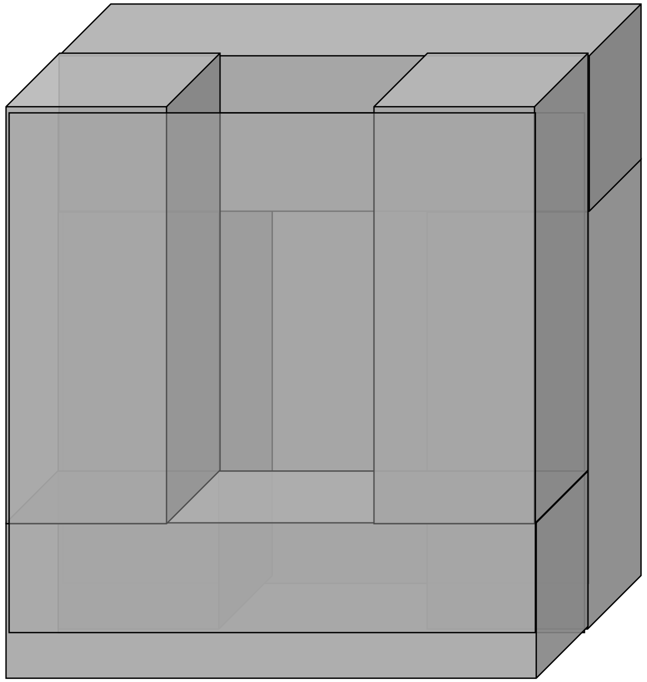


$\not\simeq$



homeomorphic but not dihomotopy equivalent

directed homotopy types

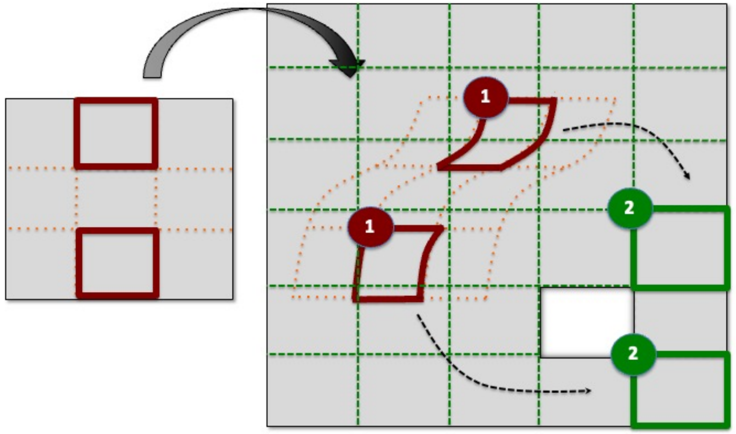


$\not\simeq \quad \bullet$

contractible but not dicontractible

[L. Fajstrup, 2009]

indecomposability of directed homotopy types



inclusion of cubes in an iterated subdivision of cubes almost never has the homotopy extension property

directed spaces do not decompose into directed homotopy colimits of simple cells

no known Whitehead Theorem

There is no known directed homotopy invariant π_* such that

$$\pi_*(f) : \pi_*(X) \to \pi_*(Y)$$

is an isomorphism if and only if f is a directed homotopy equivalence and π_* is compactly supported, for all stream maps f between streams in nature.

fundamental questions

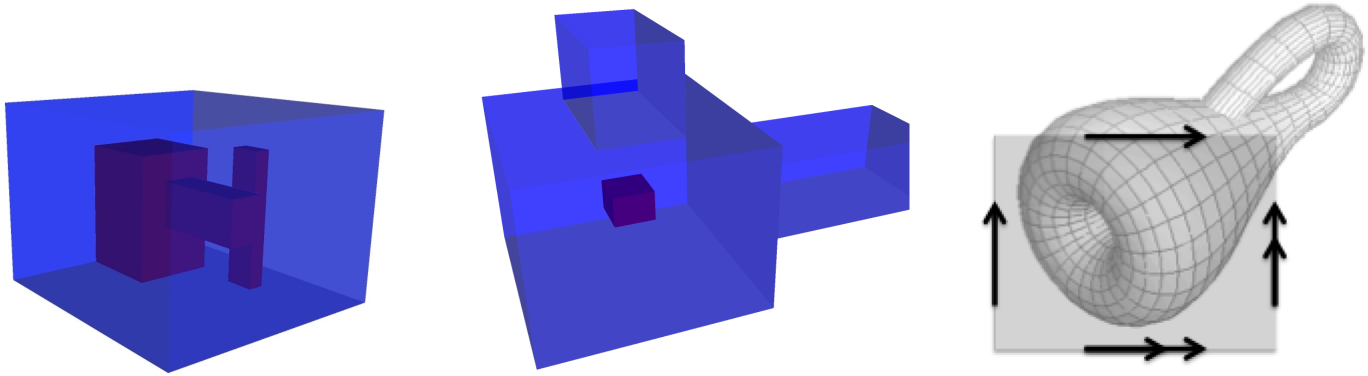
$$\hat{\bigcirc} \xrightarrow{\;|-|\;} \mathrm{Streams}$$

localization by cubical functions whose images under sing|-| are naïve combinatorial homotopy equivalences

localization by stream maps whose images under right adjoint to |-| are naïve combinatorial homotopy equivalences

$$d(\hat{\bigcirc}) \dashrightarrow d(\mathrm{Streams})$$

When do localizations exist? When is there a bottom equivalence?



III. cubical sets

working category of cubes □

minimal symmetric monoidal variant: cofaces, codegeneracies, coordinate permutations

THM TFAE for a monotone function $\varphi:[1]^m \to [1]^n$:

1. φ is a morphism in □
2. φ is an interval-preserving lattice homomorphism (between finite Boolean lattices)

cubical sets = cubical sets with symmetries = presheaves over □

[Krishnan, 2015]

examples

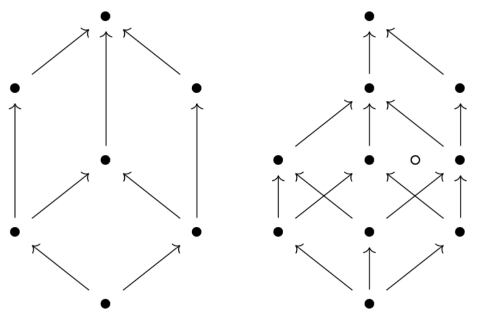
1. cubical nerves of small categories
2. singular cubical sets of streams
3. edge-oriented cubical complexes

[Krishnan, 2015]

example: Boolean intervals

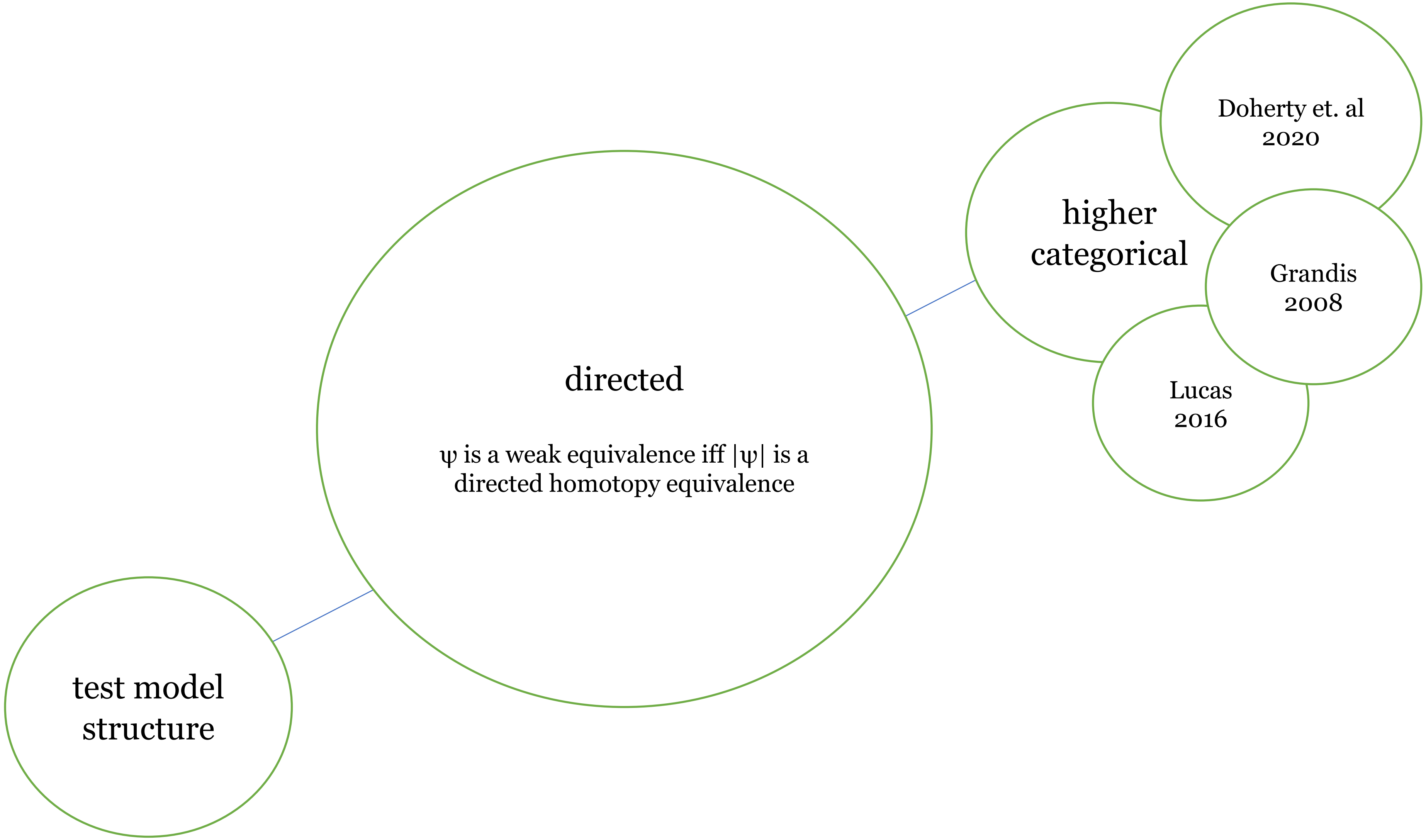
$\square[P]$ = cubical set of interval-preserving monotone functions $[1]^n \to P$

THM { isomorphism classes of finite distributive meet-semilattices } $\overset{\square[-]}{\cong}$ { isomorphism classes of finite rooted CAT(0) cubical complexes up to isometry }



[Ardila, Owen, Sullivant, 2012]

cubical homotopy theories



subdivisions

cocontinuous endofunctor sd on cSet with $\mathrm{sd}\square[1]^n = \square[2]^n = \square([1]^n)^{[1]}$

$\mathrm{sd}\square[1]^2$

convenient properties of subdivision

LEM Fix a cubical set C. There exist dotted cubical functions making

$$\begin{array}{ccc} S & \dashrightarrow & \square[1]^n \\ \downarrow & & \vdots \\ \mathrm{sd}^4 C & \longrightarrow & C \end{array}$$

commute and natural in subcubical sets S of sd^4C that lie in the closed Star of a vertex.

LEM There exist **natural** stream isomorphisms $|\mathrm{sd}\, C| \cong |C|$

[Krishnan, 2015]

cubical approximation

THM Fix cubical sets A,B **with A finite**. For each stream map

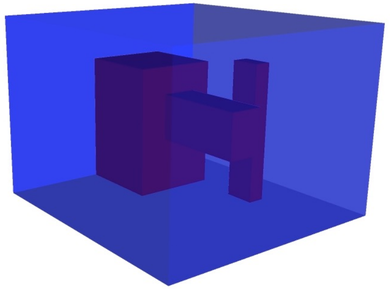
$$f:|A|\to|B|$$

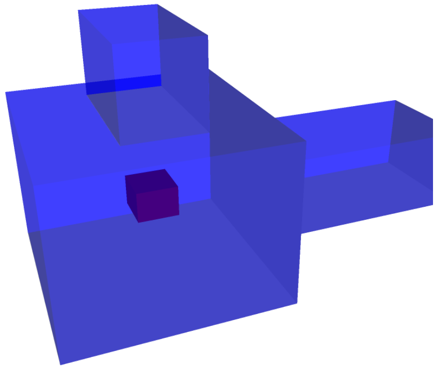
f is directed homotopic to a stream map $|A\to B|$ after replacing A with sd^kA for k sufficiently large.

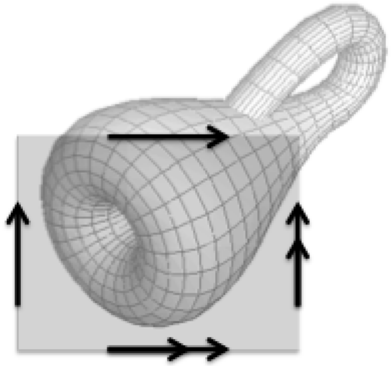
PROOF

1. After sufficiently many subdivisions, f maps cubes into stars.
2. Then f locally lifts to a directed cube.
3. Convex structure gives requisite homotopies.

[Krishnan, 2015]







IV. cubcats

definition

cubcat = cubical set C admitting structure maps

$$\operatorname{colim}_{\square[1]^n \to C} \operatorname{sing} |\square[1]^n| \xrightarrow{\nu_C} C \qquad \operatorname{ex} C \xrightarrow{\mu_C} C$$

where ex is right adjoint to sd

satisfying a compatibility condition

$$\begin{array}{ccc} C & \xrightarrow{1_C} & C \\ \downarrow & & \uparrow \mu_C \\ \operatorname{ex} \operatorname{colim}_{\square[1]^n \to C} \operatorname{sing} |\square[1]^n| & \xrightarrow[\operatorname{ex} \nu_C]{} & \operatorname{ex} C \end{array}$$

[Krishnan, 2023]

examples

THM cubical nerves are cubcats

THM singular cubical sets of streams are cubcats

THM test fibrant cubical sets are cubically homotopy equivalent to cubcats

[Krishnan, 2023]

dictionary

test fibrant cubical sets	cubcats
eg: nerves of small groupoids	eg: nerves of small categories
every cubical set is test fibrant up to classical weak equivalence	every cubical set is a cubcat up to directed weak equivalence
$h(\mathrm{cSet})(A,B) = h(\mathrm{Top})(\lvert A\rvert, \lvert B\rvert) = \pi_0 B^A$ for test fibrant B	$d(\mathrm{cSet})(A,B) = d(\mathrm{Streams})(\lvert A\rvert, \lvert B\rvert) = \pi_0 B^A$ for cubcats B

[Krishnan, 2023]

dictionary

test fibrant cubical sets

eg: nerves of small groupoids

every cubical set is test fibrant up to classical weak equivalence

$$h(\mathrm{cSet})(A,B) = h(\mathrm{Top})(|A|,|B|) = \pi_0 B^A$$

for test fibrant B

cubcats

eg: nerves of small categories

every cubical set is a cubcat up to directed weak equivalence

$$d(\mathrm{cSet})(A,B) = d(\mathrm{Streams})(|A|,|B|) = \pi_0 B^A$$

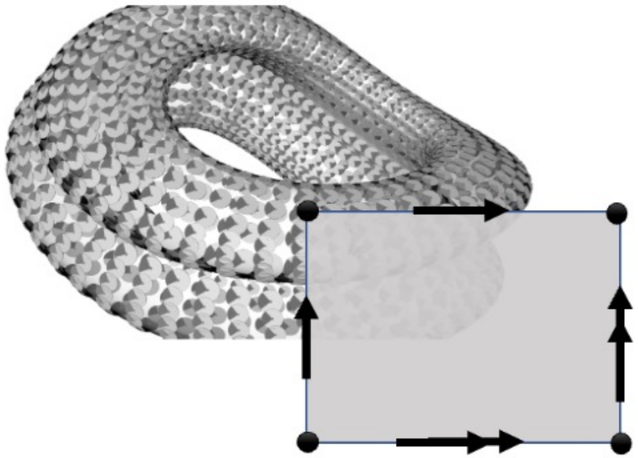
for cubcats B

requires an analogue of Reedy cofibrant replacement

[Krishnan, 2023]

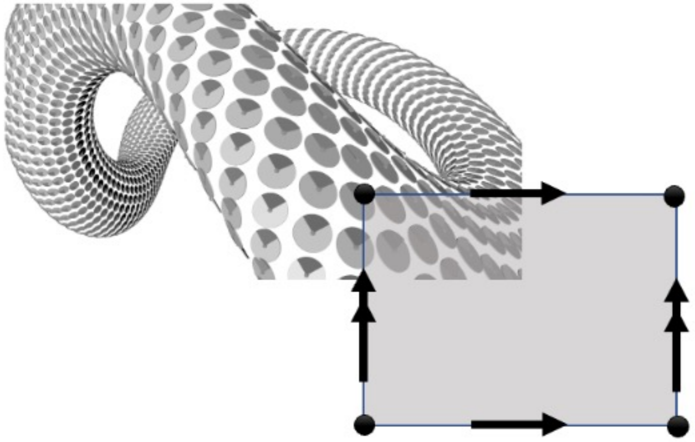
1-cohomology $H^1(-;M)=[-,|BM|]$

spacetime Klein bottle



$M \times_{2M} M$

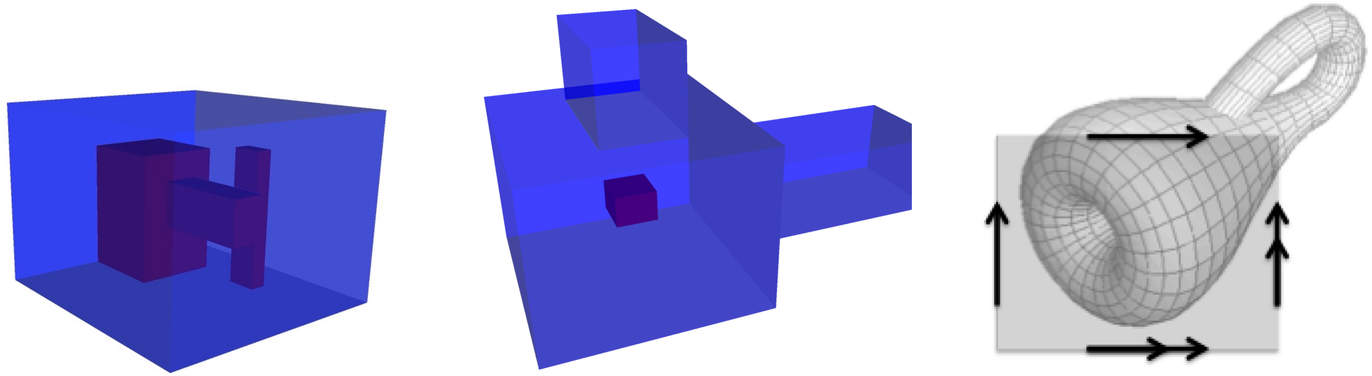
spacetime torus



$M \oplus M$

for cancellative commutative monoid coefficients M

[Krishnan, 2023]



V. big picture

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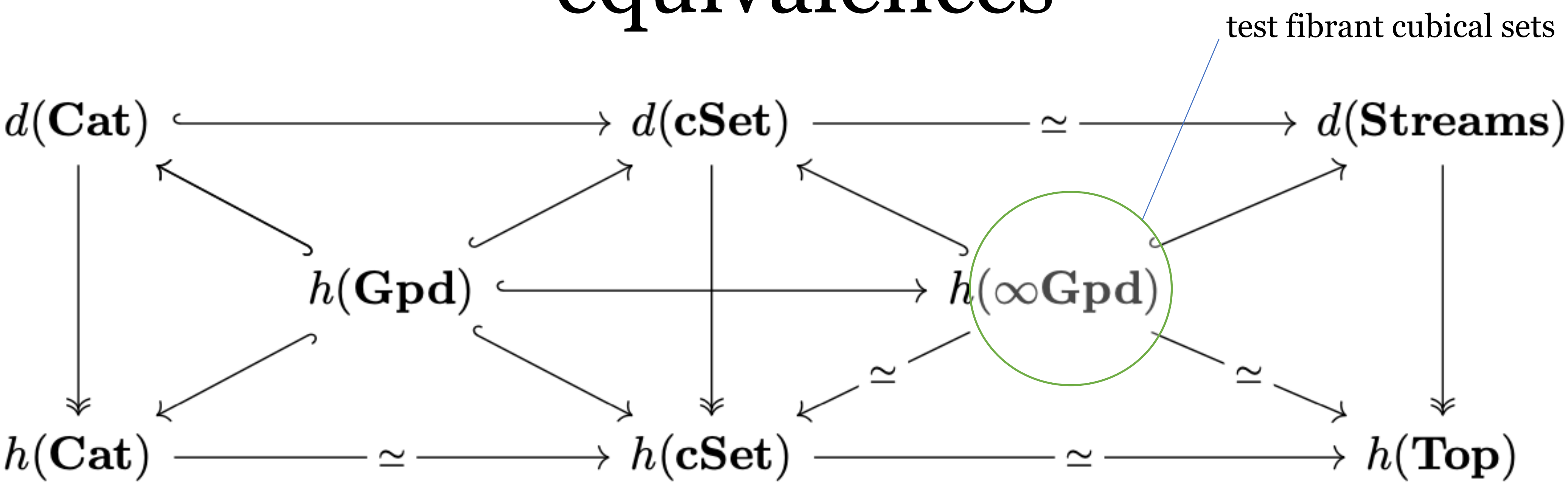
equivalences



Directed homotopy theory both refines and extends classical homotopy theory.

[Krishnan, 2023]