Proof by induction

Proof by Induction: To prove a statement $P(n)$ is true for all natural numbers n, we can do the following:

i. **Base case:** Prove it is true when $n = 1$.

ii. **Inductive step:** Prove that if $k \in \mathbb{N}$ and $P(k)$ is true, then $P(k+1)$ is also true.
The assumption that $P(k)$ is true is called the **inductive hypothesis**.

Generally, the interesting part of an inductive proof is the inductive step. However, the base case cannot be omitted—it is the first domino, and we must knock it over to start everything in motion!

Prove the following theorems using mathematical induction:

Theorem I.1. *Let n be a natural number. Then*

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Theorem I.2. *For every natural number n,*

$$1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1.$$

Theorem I.3. *For every natural number n,*

$$1 + 3 + 5 + \cdots + (2n-1) = n^2.$$

Theorem I.4. *For every natural number n,*

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Theorem I.5. *For every natural number n,*

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2 = \frac{n^2(n+1)^2}{4}.$$

Theorem I.6. *For every natural number $n > 3$, $2^n < n!$.*

Theorem I.7. *For every natural number n,*

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$

Theorem I.8 (Geometric series). *Let $r \neq 1$ be a real number. For every natural number n,*

$$1 + r + r^2 + \cdots + r^{n-1} = \frac{r^n - 1}{r - 1}.$$

Theorem I.9 (Binomial theorem). *For any non-negative integer n,*

$$(x+y)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^{n-k} y^k.$$

Here, x and y are indeterminates—you may think of them as arbitrary real numbers, or simply as variables. You may find it useful to use the notation $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Definition. The **Fibonacci numbers** are a sequence of integers, denoted F_n. The first two Fibonacci numbers are

$$F_1 = 1 \qquad \text{and} \qquad F_2 = 1.$$

For $n \geq 3$, the number F_n is defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}.$$

Exercise I.10. *List the first 10 Fibonacci numbers.*

Theorem I.11. *For every natural number n, $F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$.*

Theorem I.12. *For every natural number n, $F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}$.*

Proof by Strong Induction: To prove a statement $P(n)$ is true for all natural numbers n, we can do the following:

i. **Base case(s):** Prove it is true when $n = 1$. Depending on the nature of the proof, you may also need to prove the next few cases ($n = 2, 3 \ldots$).

ii. **Inductive step:** Prove that if $k \in \mathbb{N}$ and $P(1), P(2), \ldots, P(k)$ are all true, then $P(k+1)$ is also true.

Prove the following theorems using strong induction:

Theorem I.13. *Let $a = \dfrac{1+\sqrt{5}}{2}$ and $b = \dfrac{1-\sqrt{5}}{2}$. For every natural number n, $F_n = \dfrac{a^n - b^n}{a - b}$.*

Theorem I.14. *For every natural number n, $F_n < (5/3)^n$.*

Theorem I.15. *Every natural number greater than 7 can be written as a sum of 3's and 5's where the coefficients of 3 and 5 are nonnegative.*

Theorem I.16. *Every natural number n can be written as $n = 2^k l$ where k is a non-negative integer and l is an odd integer.*