What are the elements of a successful proof?

What are the elements of a successful presentation?

<u>Presentations</u>:
- Sign up by 3 PM
- I will email assignments
- Presentations can be live or recorded
  - Must display writing surface
  - If recorded, share link with me in advance
- Resources
  - schedule 20 min consultation with me
  - presentation buddy system
  - my office hours
- Grading scale (updated)

  ✓+ Well-rehearsed + mostly correct

  ✓ communicate the main idea(s)

  ✓- incomplete or incorrect. Not a serious attempt.

<u>Homework</u>: Think about how you will give your presentations.

Introduction to proofs

Definitions

In order to precisely state and prove theorems, we first must agree on clear **definitions** of the mathematical objects and concepts involved.

Here are two definitions of the concept of evenness:

Definition 1: An integer n is **even** if it is in the set $\{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}$.

Definition 2: An integer n is **even** if there is another integer k such that $n = 2k$.

Which of these definitions do you prefer? Why?

Def 2 - more precise, gives a recipe

Def 1 - more concrete, gives examples

Definition 2 might be rephrased as: "a number is even if it is divisible by 2." Let's make this more precise.

Definition 3: Let n and d be two integers. We say that d **divides** n, or equivalently that n **is divisible by** d if there is another integer k such that $n = dk$. In this case, we write $d \mid n$.

Mathematicians often use examples to illustrate a definition.

Example 1: The number 3 divides 12 because $12 = 3 \cdot 4$. This also shows that $4 \mid 12$. We can see that $2 \mid 12$ and $6 \mid 12$ from the expression $12 = 2 \cdot 6$. Finally, $1 \mid 12$ and $12 \mid 12$ because $12 = 1 \cdot 12$.

We can therefore conclude that

$$1, 2, 3, 4, 6, 12$$

is a complete list of positive integers which divide 12.

Example 2: If n is any integer, then $1 \mid n$ and $n \mid n$. (Why?) $n = 1 \cdot n$

Definition 4: Let $p > 1$ be an integer. We say that p is **prime** if 1 and p are the only positive integers which divide p.

What is a proof?

Mathematical knowledge is typically collected in a series of statements, called **theorems** (and also **propositions**, **lemmas**, **claims**, etc.). A theorem is a true statement which follows logically from definitions and other theorems. The reasoning which demonstrates the truth of a theorem is called a **proof**.

While theorems can be quite complicated, every theorem has the following basic structure:

If P, then Q.

Here, P stands in for a set of **hypotheses**, or assumptions, and Q stands in for a set of **conclusions**. The theorem states that whenever the hypotheses P are satisfied, then the conclusion(s) Q must be true. Your job, as a mathematician, is to provide a proof explaining why this is so!

Identify the hypotheses and conclusion in the following theorem statement.

Theorem 1: If $\underbrace{n \text{ and } m \text{ are even numbers}}_{\text{hypotheses}}$, then $\underbrace{n+m \text{ is also an even number}}_{\text{conclusion}}$.

Now, let's write a proof.

Proof: Since n is even, we can write

$$n = 2\cdot k$$

for some integer k. Similarly, m is even and so

$$m = 2\cdot \ell$$

for some integer ℓ.
Therefore,

$$n+m = 2\cdot k + 2\cdot \ell = 2(k+\ell).$$

This shows that $2 \mid (n+m)$, that is, $n+m$ is even. $\blacksquare$

Scratch work

Ex: $4 + 10 = 14$ ✓
$2\cdot 2 + 2\cdot 5 = 2\cdot 7$
$2(2+5)$

$n = 2\cdot k$
$m = 2\cdot \ell$
$\to n+m$
$= 2\cdot k + 2\cdot \ell$
$= 2(k+\ell)$

The proof we just wrote is an example of a **direct proof**, where we begin with the assumptions and work towards the conclusion. Direct proof is the most straightforward proof technique, but we will encounter others.

Direct proof

Let's practice writing some direct proofs. In each theorem statement, identify the hypotheses and conclusion.

We first introduce one more definition.

Definition 5: An integer n is **odd** if it is not even.

Theorem 2: If n is even and m is odd, then nm is even. Rooms 1-3

Proof:

Theorem 3: Let n be an integer. If $6 \mid n$, then $3 \mid n$. Rooms 4-6

Proof:

Proof by contrapositive

Suppose we wish to prove a theorem of the form "If P, then Q." We wish to show that whenever P is true, then Q must also be true. Notice that this is the same as saying that whenever Q is *false*, then P must also have been false.

In other words, the statement

$$P \implies Q$$

is logically equivalent to the statement

$$(\text{not } Q) \implies (\text{not } P).$$ "the contrapositive"

(Here, the symbol "$\implies$" is shorthand for "implies.")

Let's see this in action:

Theorem 4: Let $n > 2$ be an integer. If $\underbrace{n \text{ is prime}}_{\text{hyp.}}$, then $\underbrace{n \text{ must be odd}}_{\text{conc.}}$.

State the contrapositive of Theorem 4. Then, give a direct proof of this contrapositive statement.

Contrapositive: If n is even, then n is not prime.

Proof: Suppose n is even. Then $n = 2 \cdot k$ for some number k. Since $n > 2$, we must have $k > 1$. Therefore, n is divisible by 2 and $2 \neq 1$ and $2 \neq n$. That is, n is not prime. ∎

Do the same for the next theorem:

Theorem 5: Let n be an integer. If n^2 is odd, then n is odd.

Contrapositive:

Proof: