To prove <u>If P, then Q</u>

- Direct proof: Assume P, and work to prove Q.
- Proof by contrapositive: Prove If NOT Q, then NOT P.

E.g. If $\underbrace{n^2 \text{ is even}}_{P}$, then $\underbrace{n \text{ is even}}_{Q}$.

is equivalent to

If $\underbrace{n \text{ is odd}}_{\text{NOT } Q}$, then $\underbrace{n^2 \text{ is odd}}_{\text{NOT } P}$.

- Proof by contradiction: Assume P is true and Q is false.

Use these to derive a contradiction, i.e. a statement that can't be true. Therefore P implies Q.

In Macy's proof, if $\underbrace{r \text{ is the smallest element of the set } S}_{P}$, then $\underbrace{0 \le r \le n-1}_{Q}$.

Suppose $r \ge n$ ..... there is an element of S which is smaller than r. Contradiction.

Well-Ordering is the reason that induction works:

Thm: A statement $P(k)$ is true for all natural numbers k.

① Prove the base case: $P(1)$

② Prove the inductive step: If $P(k)$ is true, then $P(k+1)$ is true.

Claim: If you do ① and ②, then $P(k)$ is true for all natural numbers.

Proof: Suppose not. Let

$$S = \{ k \mid P(k) \text{ is false} \}.$$

If this set is not empty, then Well-Ordering implies it has a smallest element. Call it ℓ.

Then $\ell \neq 1$. ①

So $\ell - 1$ is a natural number which is smaller than ℓ, so $\ell - 1$ is not in S.

Thus, $P(\ell - 1)$ is true. So $P(\ell)$ is true. ②

Contradiction.

So S is empty. ∎

- All HW from last week returned later today.

- Exam 1 – Wednesday, 11 – 11:50 AM

  - Exam available on Canvas at 11 AM

  - ~ 40 min exam

  - submit to Canvas or my email like a HW assignment

  - Covers divisibility + congruence mod n (no division alg.) and basic proof writing + logic

  - I will be in our usual class Zoom mtg to answer questions. You don't need to join except to ask questions.

$m = 25, \quad n = 7 \longrightarrow q = 3, \quad r = 4$

$$25 = 7 \cdot \underbrace{3}_{q} + \underbrace{4}_{r}$$

$0 \le r \le 6$

Idea: q is the largest # so that
but $\boxed{\begin{array}{l} nq \le m \\ n(q+1) > m \end{array}}$

Well-Ordering: If S is non-empty set of nat. #s, then it has a smallest element.
positive integers

$$\boxed{\begin{array}{l} nq \le m \\ n(q+1) > m \end{array}} \longleftrightarrow \begin{array}{l} 0 \le m - nq \\ 0 < m - n(q+1) \end{array}$$

$$S = \left\{ m - n\cdot k \;\middle|\; \begin{array}{l} \text{where } k \text{ is an integer} \\ \text{and } m - n\cdot k \ge 0 \end{array} \right\}$$

Ex: $m = 25$, $n = 7$

$$\{25 - n\cdot 7\} = \{\ldots, 32, 25, 18, 11, 4\}$$

First: S not empty (m is in S)

Second: Well-Ordering $\rightarrow$ it has a smallest elt.
Call it r.

Now, prove
- $m - n\cdot q = r$
- $0 \le r \le n-1$ (use r is smallest)

Another approach:

$$S = \{\text{nat. numbers } k \mid nk > m-n\}$$

Ex: $n=7$, $m=25$

$$S = \{k \mid 7\cdot k > 18\}$$

$$= \{3, 4, 5, \ldots\}$$

Well-ordering: S has a smallest elt.

Call it q