Pascal's Triangle

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

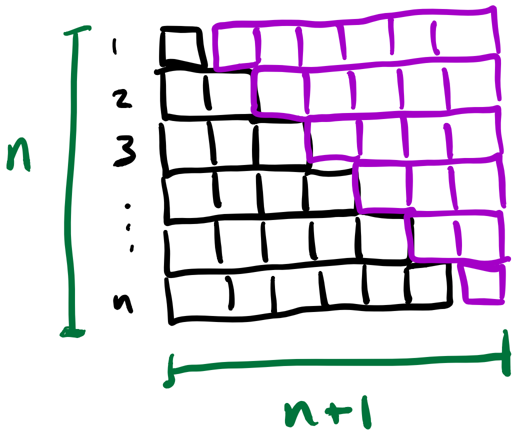
$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$1$$
$$1 \quad 1$$
$$1 \quad 2 \quad 1$$
$$1 \quad 3 \quad 3 \quad 1$$
$$1 \quad 4 \quad 6 \quad 4 \quad 1$$

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

Proof by Picture

I.1) $1+2+3+\cdots+n = \dfrac{n(n+1)}{2}$

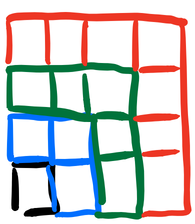


$$2(1+2+3+\cdots+n) = n(n+1)$$

I.4) $1^2+2^2+3^2+\cdots+n^2 = \dfrac{n(n+1)(2n+1)}{6}$

similar proof in 3D

I.3) $1+3+5+\cdots+(2n-1) = n^2$



Strong Induction

- Base Case(s) ✓
- Induction step: Instead of assuming $P(n)$ to prove $P(n+1)$, assume $\underbrace{P(1), P(2), \ldots, P(n)}_{\text{Inductive hypothesis}}$ to prove $P(n+1)$

Logically, this is just as good as ordinary induction

Thm: Every natural number n has a binary representation, i.e.

$$n = 2^{m_1} + 2^{m_2} + \cdots + 2^{m_s}$$

where $m_1, m_2, \ldots, m_s$ are all distinct (no repeats)

Ex: $20 = 2^4 + 2^2$

$35 = 2^5 + 2^1 + 2^0$

Proof: By strong induction on n.

Base case: $n = 1$. Then $1 = 2^0$ ✓

Inductive Step: Let n be a natural number, and assume the theorem is true for all numbers $1, 2, 3, \ldots, n$. That is, if $\boxed{1 \leq k \leq n}$, then k has a binary rep.

Want to prove: $n+1$ has a binary rep.

Let a be the largest integer such that

$$2^a \leq n+1$$

- If $2^a = n+1$, then we're done.
- Otherwise $1 \leq n+1-2^a = k \leq n$

So, by the ind. hyp.,

$$n+1-2^a = 2^{m_1} + 2^{m_2} + \cdots + 2^{m_s}$$

where the exponents are distinct

$$\Rightarrow \quad n+1 = 2^a + \underbrace{2^{m_1} + 2^{m_2} + \cdots + 2^{m_s}}_{\text{is } 2^a \text{ in here?}}$$

If it is,

$$n+1 = 2^a + 2^a + (\text{''} \quad \text{''})$$

$$= 2 \cdot 2^a + \cdots$$

$$= 2^{a+1} + \cdots$$

$$n+1 - 2^{a+1} = \cdots \geq 0$$

$$n+1 \geq 2^{a+1}$$

which contradicts how we chose a.

So there are no repeated exponents. $\blacksquare$