If $ax_0 + by_0 = c$

then $x = x_0 + k \cdot \frac{a}{(a,b)}$, $y = y_0 - k \cdot \frac{b}{(a,b)}$

is also a solution of $ax + by = c$.

Moreover, these are all of the solutions.

Suppose I have a solution x, y.

$$ax + by = c.$$

Since

$$ax_0 + by_0 = c,$$

we have

$$ax + by = ax_0 + by_0$$

$$ax - ax_0 = by_0 - by$$

$$a(x - x_0) = b(y_0 - y)$$

Then

$$\frac{a(x - x_0)}{(a,b)} = k = \frac{b(y_0 - y)}{(a,b)}$$ is an integer

Lemma: $\frac{a}{(a,b)}$ and $\frac{b}{(a,b)}$ are relatively prime

Proof: $\frac{a}{(a,b)} = l_1 \qquad \frac{b}{(a,b)} = l_2$

$$a = l_1 (a,b) \qquad b = l_2 (a,b)$$

If $d > 0$ and $d \mid l_1$ and $d \mid l_2$, then

$$a = (d \cdot m_1) \cdot (a,b) \qquad b = (d \cdot m_2) \cdot (a,b)$$

$\Rightarrow$ $d \cdot (a,b)$ is a common divisor of a and b.

$d \cdot (a,b) \leq (a,b) \Rightarrow d = 1.$ ✓

So

$$\frac{a(x - x_0)}{(a,b)} = \frac{b(y_0 - y)}{(a,b)}$$ is an integer

$$\frac{a}{(a,b)} \cdot (x - x_0) = \frac{b}{(a,b)} \cdot (y_0 - y)$$

$\Rightarrow$ $\frac{b}{(a,b)}$ divides $\frac{a}{(a,b)} \cdot (x - x_0)$ and is rel. prime to $\frac{a}{(a,b)}$

(if $a \mid bc$ an $(a,c)=1$, then $a \mid c$)

By applying theorem 1.41 to this equation, we see

Thm 1.41 $\Rightarrow$ $\dfrac{b}{(a,b)}$ divides $(x-x_0)$

$$\Rightarrow \quad \frac{x-x_0}{\left(b/(a,b)\right)} = (x-x_0)\frac{(a,b)}{b} \text{ is an integer.}$$

Same reasoning: $\dfrac{y_0-y}{\left(a/(a,b)\right)} = (y_0-y) = \dfrac{(a,b)}{a}$ is an integer

$$\frac{a(x-x_0)}{(a,b)} = \frac{b(y_0-y)}{(a,b)}$$

$$\rightarrow \quad \frac{(a,b)}{b}\cdot(x-x_0) = \frac{(a,b)}{a}(y_0-y) = k$$

Then

$$\frac{(a,b)}{b}(x-x_0) = k \rightarrow x-x_0 = k\cdot\frac{b}{(a,b)}$$

$$x = x_0 + k\cdot\frac{b}{(a,b)}$$

$$\frac{(a,b)}{a}(y_0-y) = k \rightarrow y_0-y = k\cdot\frac{a}{(a,b)}$$

$$y = y_0 - k\cdot\frac{a}{(a,b)}$$

$\gcd(12, 28) = 4$

$\gcd(12\cdot 5, 28\cdot 5) = \gcd(60, 140) = 4\cdot 5 = \underline{20}$

$5\cdot 12x + 5\cdot 28y = c$ an integer

$= 5\cdot k$

$\rightarrow\ 12x + 28y = k$ ← the smallest positive # that can go here is $(12,28)=4$

1.541 $\quad 24x + 9y = 33$

By the Euclidean algorithm,

$24 = 2\cdot 9 + 6 \rightarrow 24 - 2\cdot 9$

$9 = 1\cdot 6 + 3 \rightarrow 3 = 9 - 6$

$6 = 2\cdot 3$

$3 = 9 - 6 = 9 - (24 - 2\cdot 9)$

$= 24(-1) + 9(3)$

$24(\underline{-11}) + 9(\underline{33}) = 33$

$$\boxed{\begin{aligned} x_0 &= -11 \\ y_0 &= 33 \end{aligned}}$$

$$x = -11 + k \cdot \frac{9}{(24,9)} \qquad (24,9) = 3$$

$$= -11 + k \cdot 3$$

$$y = 33 - k \cdot \frac{24}{(24,9)}$$

$$= 33 - k \cdot 8$$

All solutions

	x	y	
			$k=-2$
			$k=-1$
	-11	33	x_0, y_0
	-8	25	$k=1$
	-5	17	$k=2$
$k=-1$	-2	9	$k=3$
x_0, y_0	1	1	$k=4$
$k=1$	4	-7	$k=5$