Office Hours

MW 3-4 pm

Thur 1-2 pm

As always, I'm available by appointment.

Lemma 2.8: Let $p, q_1, \ldots, q_n$ be primes. If $p \mid q_1 \cdots q_n$, then $p = q_i$ for some i.

Non-Ex: $p = 4$, $q_1 = 6$, $q_2 = 9$, $q_3 = 10$

$4 \mid (6 \cdot 9 \cdot 10)$ $\qquad 540 = 4 \cdot 135$

but $4 \neq 6, 9,$ or 10

Proof: By induction on n.

Base case: $n=1$. Then $p \mid q_1$, so $pk = q_1$ for some integer k. Since q_1 is prime, its only factorization is

$$q_1 = 1 \cdot q_1 .$$

Since p is prime, $p \neq 1$, so

$$p = q_1 \quad (\text{and } k = 1).$$

Inductive step: Assume that if p divides a product of n primes, then p is equal to one of those primes.

Now suppose

$$p \mid q_1 \cdots q_n \cdot q_{n+1}$$

Case 1: $p = q_1$, then we're done.

Case 2: $p \neq q_1$ then $(p, q_1) = 1$.

By Thm 1.41, $p \mid q_1 \cdot (q_2 \cdots q_{n+1})$,

so $p \mid \underbrace{q_2 \cdots q_{n+1}}_{n \text{ primes}}$.

By the inductive hypothesis, $p = q_i$ for one of the i in $2 \le i \le n+1$. $\blacksquare$

Thm 2.9 (FTA, uniqueness part)

Let n be a natural number,

$$n = p_1^{r_1} p_2^{r_2} \cdots p_m^{r_m} = q_1^{t_1} q_2^{t_2} \cdots q_s^{t_s}$$

where

- $\{p_1, \ldots, p_m\}$ are a set of distinct primes

  $p_i \neq p_j$ if $i \neq j$

- $\{q_1, \ldots, q_s\}$ also a set of distinct primes.

Then $m = s$, $\{p_1, \ldots, p_m\} = \{q_1, \ldots, q_s\}$, and if $p_i = q_j$ then $r_i = t_j$.

Proof: Strong induction on n.

Base case: $n=1$, nothing to do

Inductive step: Assume every natural number k satisfying $1 \le k \le n$ has a unique prime factorization.

We'll prove $n+1$ has a unique prime factorization.

By theorem 2.1, there exists a prime p such that $p \mid (n+1)$. $\Rightarrow n+1 = p \cdot k$

If

$$n+1 = p_1^{r_1} p_2^{r_2} \cdots p_m^{r_m} = q_1^{t_1} q_2^{t_2} \cdots q_s^{t_s}$$

By Lemma 2.8, p divides the product of p_i's, so $p = p_i$ for some i. Similarly, $p = q_j$ for some j.

$$\Rightarrow n+1 = p \cdot k$$

$$\Rightarrow k = p_1^{r_1} \cdots p_i^{r_i - 1} \cdots p_m^{r_m}$$

$$= q_1^{t_1} \cdots q_j^{t_j - 1} \cdots q_s^{t_s}$$

By the inductive hypothesis, these are the same prime factorization of k

$\Rightarrow m = s$

$\{p_1, \dots, p_m\} = \{q_1, \dots, q_s\}$

and

the exponents agree. $\blacksquare$