1 Recall that $\mathrm{GL}_n(\mathbb{R})$ is the group of all $n \times n$ invertible matrices with entries in $\mathbb{R}$. Recall also that the **determinant** is a function $\det\colon \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}^\times$ with the properties that

- $\det(AB) = \det(A)\det(B)$ for all $A, B \in \mathrm{GL}_n(\mathbb{R})$;
- $\det(I_n) = 1$, where $I_n \in \mathrm{GL}_n(\mathbb{R})$ is the $n \times n$ identity matrix.

Prove that the **special linear group**

$$\mathrm{SL}_n(\mathbb{R}) = \{A \in \mathrm{GL}_n(\mathbb{R}) \mid \det(A) = 1\}$$

is a normal subgroup of $\mathrm{GL}_n(\mathbb{R})$.

2 Let $H = \{1, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\} \leq S_4$. (You should convince yourself that H is in fact a subgroup of S_4; however, you do not need to prove it here.)

(a) Prove that $H \trianglelefteq S_4$.

(b) What is the cardinality of S_4/H? Write a complete list (with no repetitions) of the cosets of H in S_4.

(c) Write the group operation table for the quotient group S_4/H.

(d) Give an example of a homomorphism with domain S_4 and kernel H.

3 Let G be a group and let N be a normal subgroup of G. Prove that the cosets xN and yN commute in G/N if and only if $x^{-1}y^{-1}xy \in N$. (The element $x^{-1}y^{-1}xy$ is called the **commutator** of x and y and is denoted by $[x, y]$.)

4 Let G be a group. Prove that

$$C = \langle x^{-1}y^{-1}xy \mid x, y \in G \rangle$$

is a normal subgroup of G, and that G/C is abelian. (C is called the **commutator subgroup** of G.)