1 Let G be a group acting on a set X. Let $\sim$ be the relation on X defined by

$$x_1 \sim x_2 \quad \text{if and only if} \quad \text{there exists } g \in G \text{ such that } gx_1 = x_2.$$

Prove that $\sim$ is an equivalence relation on X.

2 Let $G = \mathrm{GL}_2(\mathbb{R})$ be the group of invertible 2×2 matrices with entries from $\mathbb{R}$ (with matrix multiplication as the group operation). Let

$$X = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \;\middle|\; x, y \in \mathbb{R} \right\}$$

Define a function $G \times X \to X$ by matrix multiplication:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}.$$

(a) Prove that this is a group action.

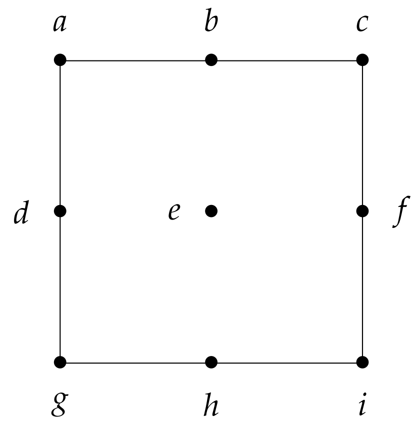
(b) Prove that this action is faithful.

(c) Prove that this action is *not* transitive.

3 The group

$$D_4 = \langle r, s \mid r^4 = s^2 = 1, rs = sr^{-1} \rangle$$

acts on the square below, where r acts as a 90° clockwise rotation and s acts as a reflection across the axis of symmetry passing through the vertex labelled a.



We consider the action of D_4 on the set $X = \{a, b, c, \ldots, i\}$ of nine labelled points.

(a) Find all of the orbits of the action of D_4 on X.

(b) Find the **stabilizer subgroup** $G_a = \{g \in D_4 \mid g \cdot a = a\}$ of the point a.

(c) Find the stabilizer subgroup G_b of the point b.

(d) Find the **fixed points** of the element r. That is, find all $x \in X$ such that $r \cdot x = x$.

(e) Find the fixed points of the element sr.