1 Let G be a finite group acting on a finite set X.

(a) Suppose $|G| = 3^4 = 81$. Prove that if 3 does not divide $|X|$, then there is some $x \in X$ such that $G_x = G$.

(b) More generally, suppose $|G| = p^k$, where p is a prime number and k is a positive integer. Prove that if p does not divide $|X|$, then there is some $x \in X$ such that $G_x = G$.

2 Let G be a finite group acting transitively on a set X.

(a) Show that $|X|$ divides $|G|$. In particular, X is a finite set.

(b) Use Burnside's Lemma to prove that there is at least one group element with no fixed points; that is, there is an element $g \in G$ such that X^g is the empty set.

3 The four edges of a square are to be painted. We have 6 colors of paint available. Only one color is used on each edge, and the same color may be used on multiple edges.

(a) Let X be the set of all ways to paint the four edges. Compute $|X|$.

(b) Let D_4 act on X in the natural way. For each element $g \in D_4$, compute the number $|X^g|$ of colorings fixed by g.

(c) Use Burnside's Lemma to compute the number of *distinguishable* ways the square can be painted.