**1** Let $\mathbb{Z}[x]$ be the ring of polynomials in the variable $x$ with integer coefficients. Determine which of the following are ideals in $\mathbb{Z}[x]$. Provide justification for your answers.

(a) The set of all polynomials whose constant term is odd.

(b) The set of all polynomials whose constant term is even.

(c) The set of all polynomials whose coefficient of $x^2$ is even.

(d) The set of all polynomials whose constant term, coefficient of $x$, and coefficient of $x^2$ are zero.

(e) The set of all polynomials $p(x)$ such that $p'(0) = 0$, where $p'(x)$ is the usual first derivative of $p(x)$ with respect to $x$.

**2** Let $E$ and $F$ be fields, and let $\varphi\colon E \to F$ be a homomorphism.

(a) Prove that if $\varphi(1) = 0$, then the image $\varphi(E)$ is equal to $\{0\}$.

(b) Prove that if $\varphi(1) \neq 0$, then $\varphi(1) = 1$.

(c) Prove that if $\varphi(1) \neq 0$, then $\varphi$ is injective.

**3** Let $\varphi\colon R \to S$ be a ring homomorphism.

(a) Prove that if $J$ is an ideal of $S$, then $\varphi^{-1}(J)$ is an ideal of $R$.

(b) Suppose $R$ is a subring of $S$ and $\varphi$ is the inclusion homomorphism. Use part (a) to prove that if $J$ is an ideal of $S$, then $J \cap R$ is an ideal of $R$.

(c) Give an example where $I$ is an ideal of $R$, but $\varphi(I)$ is not an ideal of $S$.

(d) Prove that if $\varphi$ is surjective and $I$ is an ideal of $R$, then $\varphi(I)$ is an ideal of $S$.