1

(a) Define a binary operation $*$ on $\mathbb{Q}$ by

$$a * b = a + b - ab.$$

Answer the following questions

i. Is $*$ associative?

ii. Is $*$ commutative?

iii. Is there an identity element for $*$?

iv. Which elements of $\mathbb{Q}$ have an inverse with respect to $*$?

In each case, justify your answer with a brief proof or explanation.

(b) Define a binary operation $\star$ on $\mathbb{Q}$ by

$$a \star b = \frac{ab}{3}.$$

Answer questions (i.)–(iv.) for the binary operation $\star$.

2 Let

$$H = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\} \subset M_2(\mathbb{R}).$$

(a) Show that H is closed under matrix addition and matrix multiplication.

(b) Show that $\langle \mathbb{C}, + \rangle$ is isomorphic to $\langle H, + \rangle$.

(c) Show that $\langle \mathbb{C}, \cdot \rangle$ is isomorphic to $\langle H, \cdot \rangle$.

This shows that H is a **matrix representation** of the complex numbers $\mathbb{C}$.

3 Let $G = \{z \in \mathbb{C} \mid z^n = 1 \text{ for some } n \in \mathbb{Z}^+\}$.

(a) Prove that G is closed under multiplication.

(b) Show that G is *not* closed under addition.

(c) Prove that G is a group under multiplication. This group is called the group of **roots of unity** in $\mathbb{C}$.